CHAPTER 8

SPECTRUM ANALYSIS

INTRODUCTION

We have seen that the frequency response function $T(j\omega)$ of a system characterizes the amplitude and phase of the output signal relative to that of the input signal for purely harmonic (sine or cosine) inputs. We also know from linear system theory that if the input to the system is a sum of sines and cosines, we can calculate the steady-state response of each sine and cosine separately and sum up the results to give the total response of the system. Hence if the input is:

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{k=10} B_k \sin \; \omega_k t - \theta_k \qquad (1)$$

then the steady state output is:

$$y(t) = \frac{A_0}{2} T(j0) + \sum_{k=1}^{k=10} B_k \left|T(j\omega_k)\right| \sin \; \omega_k t - \theta_k + \angle T(j\omega_k) \qquad (2)$$

Note that the constant term, a term of zero frequency, is found from multiplying the constant term in the input by the frequency response function evaluated at $\omega = 0$ rad/s.

So having a sum of sines and cosines representation of an input signal, we can easily predict the steady state response of the system to that input. The problem is how to put our signal in that sum of sines and cosines form. For a periodic signal, one that repeats exactly every, say, T seconds, there is a decomposition that we can use, called a Fourier Series decomposition, to put the signal in this form. If the signals are not periodic we can extend the Fourier Series approach and do another type of spectral decomposition of a signal called a Fourier Transform. In this chapter much of the emphasis is on Fourier Series because an understanding of the Fourier Series decomposition of a signal is important if you wish to go on and study other spectral techniques.

This Fourier theory is used extensively in industry for the analysis of signals. Spectrum analyzers that automatically calculate many of the functions we discuss here are readily available from hardware and software companies. See for example, the advertisements in the IEEE Signal Processing Magazine. Spectral analysis is popular because examination of the frequency content in a signal is often useful when trying to understand what physical components are contributing to a signal. Physical quantities, such as machine rotation rates, structural resonances and effects of material treatments, often have an easily recognizable effect on the frequency representation of the signal. The blade passage rate of rotors and fans in helicopters and turbomachinery will show up as a series of peaks in the spectrum at multiples of the blade passage frequency. Resonance phenomena, that can be related to natural frequencies of plates, beams and shells or of acoustical spaces in machines, will show up as elevated regions in the spectrum. Damping material in an acoustic space will give rise to a high frequency roll off in the spectrum, and a broadening of resonance phenomena.

In this chapter, we consider briefly three types of signals:

1. *Periodic Signals*

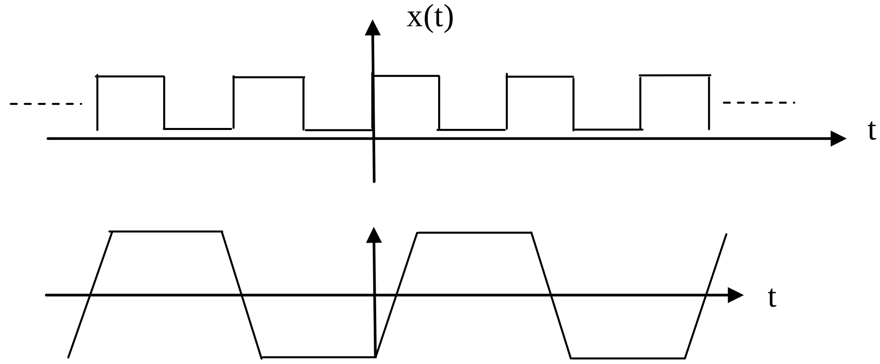


Figure 1: Examples of periodic signals.

Periodic signals repeat themselves exactly, and are observed in practice after a machine or process has been turned on and has reached steady state, i.e., any initial transient has died out. Analysis of such signals is accomplished by use of Fourier Series. Examples of simple mathematical signals that are periodic are sines, cosines and square waves. Examples of periodic signals encountered in practice include vibration of rotating machines operating at a constant speed, engine noise at constant rpm, and sustained notes on musical instruments.

2. *Well Defined Non-Periodic Signals*

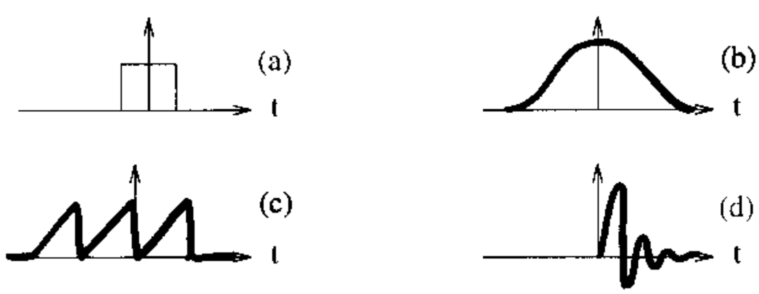


Figure 2*:* Examples of transient signals.

These signals may be repetitive as in the one shown in Figure 2(c), but only over a finite interval. These signals are analyzed by means of the Fourier Transform. In practice transients are seen when components interact, such as a valves closing, or worn, non spherical ball bearings impacting, or engines firing, or buildings responding to earthquakes or structures responding to explosions, or punch press noise.

3. *Random (Stochastic) Signals*

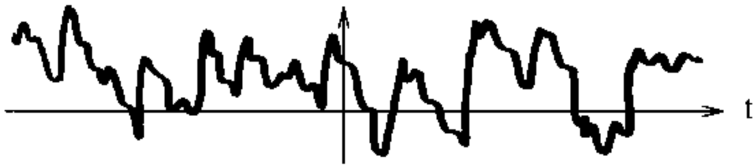


Figure 3*:* An example of a random (stochastic) signal

Random signals must be treated statistically, whereby we talk about the average properties of the signal. One commonly calculated function is the power spectral density of a signal (PSD). The power spectral density shows how the average power of the signal is distributed across frequency. We will not go into this in any detail here. However, the material presented in these notes will provide a general understanding of how a system will respond to such signals. Examples of random signals are air-movement noise in HVAC systems, motion of particles in sprays, electronic noise in measurements, and turbulent fluid motion.

As stated above, use of Fourier analysis is very common in industry. One application is machinery condition monitoring. The growth of frequency components in the spectrum over time, is often used to detect wear in components such as gears and bearings. We also use Fourier analysis to gain understanding of the signal generation. It is important to remember that the measured signal (time history) and its spectrum are two pictures of the same information. You will want to look at both representations of the signal, when trying to analyze where the primary contributions to the signal are coming from. Under some circumstances it is easier to extract information from the time history, for example, timing and level of impacts which may be important when assessing possible damage to a system. Under other circumstances, more insight is gained from observation of the spectrum, i.e., the signal decomposed as a function of frequency.

We use the Fourier series decomposition of a signal here, to enable us to predict the steady state response of a measurement system to a complicated periodic input. We are primarily interested in seeing how the measurement system distorts the signal. However, this is not the only use of Fourier analysis. In addition to those applications mentioned above, Fourier series are also used to find approximate solutions to differential equations when closed form solutions are not possible. Another application of Fourier analysis is the synthesis of sounds such as music, or machinery noise.

Following is an introduction to Fourier Series, Fourier Transforms, the Discrete Fourier Transform (for calculation of Fourier Series coefficients with a computer) and ways of describing the spectral content of random signals.

PERIODIC SIGNALS AND FOURIER SERIES ANALYSIS

Fourier series is a mathematical tool for representing a periodic function of period T, as a summation of simple periodic functions, i.e., sines and cosines, with frequencies that are integer multiples of the fundamental frequency, $\omega_1 = 2\pi f_1 = 2\pi / T$ rad/s. The kth frequency component is:

$$\omega_k = k\, 2\pi f_1 = k\, \omega_1 = \frac{2\pi k}{T} \text{ rad/s} \tag{3}$$

A picture of a periodic function is shown in Figure 4. A Fourier series expansion can be made for any periodic function which satisfies relatively simple conditions: the function should be piecewise continuous and a right and left hand derivative exist (be finite) at every point.

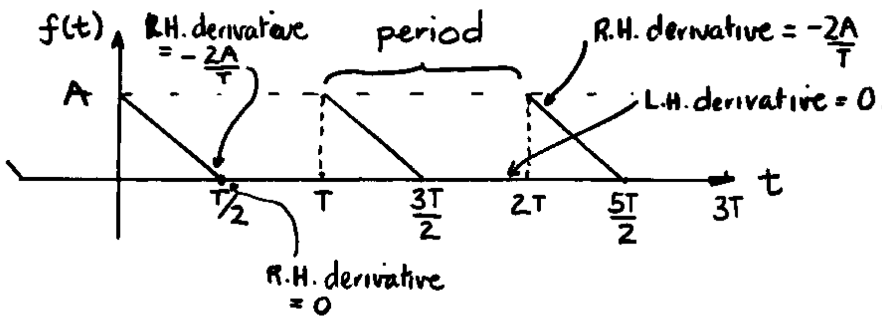


Figure 4: An illustration of the main features of a periodic function

There are several forms of the Fourier series. In this measurements course our functions are usually signals that are functions of time, which we denote by, e.g., x(t). One commonly used form of the Fourier series is where the signal is expressed as a sum of sines and cosines without phase shifts,

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos k\omega_1 t + B_k \sin k\omega_1 t \tag{4}$$

where: $\omega_1 = 2\pi / T$, is the fundamental frequency (rad/sec) and T is the period,

$A_0 / 2$ is the amplitude of the zero frequency (D.C.) component,

A_k, B_k are the Fourier coefficients,

$k\omega_1$ is the k^{th} harmonic (integer multiple of the fundamental frequency).

The Fourier coefficients A_0, A_k, B_k are defined by the integrals,

$$A_0 = \frac{2}{T} \int_0^T x(t)\, dt \tag{5a}$$

$$A_k = \frac{2}{T} \int_0^T x(t) \cos k\omega_1 t\, dt \quad k = 1, 2, \ldots \tag{5b}$$

$$B_k = \frac{2}{T}\int_0^T x(t)\sin k\omega_1 t\,dt \quad k = 1,2,\ldots \tag{5c}$$

Plotting the Fourier Series Coefficients: Amplitude and Phase Spectra

To plot the Fourier series coefficients we combine the A_k and B_k the into an amplitude and phase form. In effect, we use another representation of the Fourier Series to generate an amplitude and phase. Since a sine wave can be expressed as a cosine wave with a phase shift (or vice versa). It is possible to express the Fourier series expansion in the form shown below:

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} M_k \cos(k\omega_1 t - \theta_k) \tag{6}$$

where $M_k = \sqrt{A_k^2 + B_k^2}$ and $\theta_k = \arctan\left(\frac{B_k}{A_k}\right)$ (7a and b)

The relationship between the A_k and B_k and the M_k and θ_k can be derived by expanding the cosine with the phase shift, using trigonometrical identities, and comparing the result to the kth term in the sine and cosine form of the Fourier Series.

$$M_k \cos(k\omega_1 t)\cos(\theta_k) + M_k \sin(k\omega_1 t)\sin(\theta_k) = A_k \cos(k\omega_1 t) + B_k \sin(k\omega_1 t)$$

From this it can be seen that:

$$A_k = M_k \cos(\theta_k) \quad \text{and} \quad B_k = M_k \sin(\theta_k) \tag{8a and b}$$

Hence, the results shown above.

An equivalent expansion in terms of only sine waves can also be made.

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} M_k \sin(k\omega_1 t + \Psi_k) \tag{9}$$

where $M_k = \sqrt{A_k^2 + B_k^2}$ and $\Psi_k = \arctan\left(\frac{A_k}{B_k}\right)$ (10a and b)

So to plot the amplitude and phase spectra, we plot M_k versus $\omega_k = k\omega_1$ (amplitude spectrum), and we plot θ_k versus ω_k (phase spectrum). This is illustrated in the example shown below.

Example

Consider the periodic rectangular pulse train signal shown in Figure 5. Calculate the Fourier Series coefficients (A_k, B_k and $A_0/2$). Plot the amplitude and phase spectra of the signal.

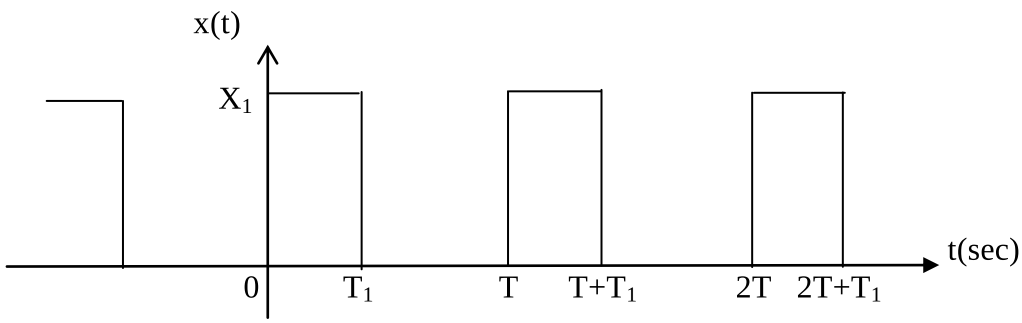


Figure 5: Rectangular pulse train signal of period T, pulse width = T,

$$x(t) = X_1 \quad 0 \le t \le T_1$$
$$\qquad 0 \quad T_1 < t < T$$

Solution

The coefficients in this case are:

$$\frac{A_0}{2} = \frac{X_1 T_1}{T} \;,\; A_k = \frac{X_1}{k\pi}\left[\sin\frac{2\pi k T_1}{T}\right] \;,\; B_k = \frac{X_1}{k\pi}\left[1 - \cos\frac{2\pi k T_1}{T}\right]$$

See details of these calculations in the section on Examples of Fourier Series, or try calculating these yourself from the formulae for A_0, A_k and B_k above.

To plot the amplitude spectrum calculate $M_k = \sqrt{A_k^2 + B_k^2}$ and plot this versus $k\omega_1$, the frequency of the kth component.

To plot the phase spectrum, calculate $\theta_k = \tan^{-1}(B_k/A_k)$.
If you are doing this in a program use **atan2**(B_k, A_k) so that the result will be in the range $\pm\,\pi$ rather than $\pm\,\pi/2$ radians.

Since we only have values to plot at discrete frequency points: $\omega = k\omega_1$, for k = 1,2,3...., the spectra are a series of lines, and hence are often called line *spectra.*

(In MATLAB the program **stem** should be used instead of **plot** to produce these line spectra.)

Sometimes the spectra are plotted against ω_k rad/s and other times they are plotted against $f_k = \omega_k/2\pi$ Hz. The normalized amplitude M_k / X_1 and θ_k are plotted in Figure 6 for the case where

$T_1 = T/4$ and $T = 0.125$ seconds. Here the amplitude and phase of the coefficients are plotted versus frequency in Hertz.

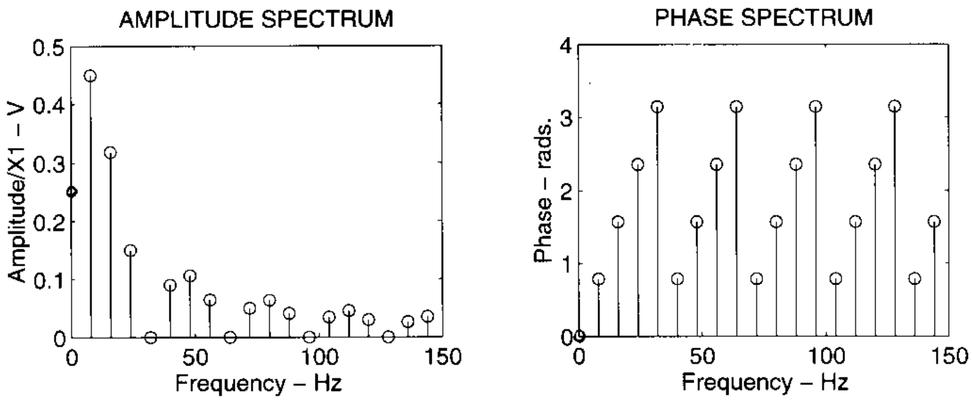


Figure 6: Line spectra for the signal shown in Figure 5.

The MATLAB m-file to do this plot is listed below.

```
% ch8f6.m program to plot the Fourier Coefficients
%            of a pulse train. T1=T/4 and T=0.125second.
%
X1=1; T=0.125; T1=T/4;
A0_2=X1*T1/T;
k=1:18;
Ak=X1*sin(2*pi*k*T1/T)./(k*pi);
Bk=X1*(1-cos(2*pi*k*T1/T))./(k*pi);
Thk=atan2(Bk,Ak);
Mk=sqrt(Ak.*Ak+Bk.*Bk);
fk=k/T;
subplot(221)
stem([0 fk],[A0_2 Mk])
xlabel('Frequency – Hz')
ylabel('Amplitude/X1 – V')
title('AMPLITUDE SPECTRUM')
subplot(222)
stem([0 fk],[0 Thk])
xlabel('Frequency – Hz')
ylabel('Phase - rads.')
title('PHASE SPECTRUM')
```

The first few terms in the Fourier Series expansion are:

$$x(t) = \frac{X_1}{4} + \frac{X_1}{\pi}\cos\omega_1 t - \frac{X_1}{3\pi}\cos 3\omega_1 t + \ldots$$

$$+\frac{X_1}{\pi}\sin\omega_1 T+\frac{X_1}{\pi}\sin 2\omega_1 t+\frac{X_1}{3\pi}\sin 3\omega_1 t+\ldots$$

The Complex Form of the Fourier Series

We derive this by considering the sine and cosine form of the Fourier Series.

$$x(t)=\frac{A_0}{2}+\sum_{k=1}^{\infty}A_k\cos k\omega_1 t+B_k\sin k\omega_1 t \tag{11}$$

where $\omega_1 = 2\pi/T$ and T is the period.

Using Euler's expansion, we can expand the sines and cosines into a sum of two complex exponentials.

$$\cos k\omega_1 t=\frac{1}{2}\left[e^{jk\omega_1 t}+e^{-jk\omega_1 t}\right] \text{ and } \sin k\omega_1 t=\frac{1}{2j}\left[e^{jk\omega_1 t}-e^{-jk\omega_1 t}\right] \tag{12}$$

Note that we are using the notation: $j=\sqrt{-1}$ and hence $\frac{1}{j}=-j$.

Substitution into the Fourier series representation above gives:

$$x(t)=\frac{A_0}{2}+\sum_{k=1}^{\infty}\left[\frac{1}{2}(A_k-jB_k)e^{jk\omega_1 t}+\frac{1}{2}(A_k+jB_k)e^{-jk\omega_1 t}\right] \tag{13}$$

Let

$$c_0=\frac{A_0}{2},\quad c_k=\frac{1}{2}(A_k-jB_k),\ \text{and}\quad c_{-k}=\frac{1}{2}(A_k+jB_k) \tag{14}$$

then

$$x(t)=c_0+\sum_{k=1}^{\infty}\left[c_k e^{jk\omega_1 t}+c_{-k}e^{-jk\omega_1 t}\right] \tag{15}$$

or

$$x(t)=c_0+\sum_{k=1}^{\infty}c_k e^{jk\omega_1 t}+\sum_{k=-1}^{-\infty}c_k e^{jk\omega_1 t} \tag{16}$$

or

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_1 t} \quad , \tag{17}$$

and the coefficients can be calculated using:

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_1 t}\, dt \quad . \tag{18}$$

This is the complex form of the Fourier series. Note:

- the DC term is c_0 and is the k=0 term,
- there are positive frequency terms (k > 0) and negative frequency terms (k < 0),
- c_{-k} is the complex conjugate of c_k.

The A_k and B_k contain information from c_k and c_{-k}. When we plot the amplitude and phase spectra, after calculating c_k, we usually plot $|c_k|$ versus ω_k as the amplitude spectrum and $\angle c_k$ versus ω_k as the phase spectrum. Note this is not quite the same as plotting M_k and θ_k. From the derivation above, it is possible to show:

$$|c_k| = \frac{1}{2}\left(A_k^2 + B_k^2\right)^{1/2} = \frac{1}{2} M_k \ , \ \text{and} \ \ \angle c_k = \tan^{-1}\frac{-B_k}{A_k} = -\theta_k \tag{19}$$

Also,

$$A_k = 2\,\text{Real}(c_k) \quad \text{and} \quad B_k = -2\,\text{Imaginary}(c_k) \tag{20}$$

Example

Consider the simple periodic function

x(t) = 9 Volts for 0<t<2 seconds and x(t) = 0 Volts for 2<t<4

x(t+q4) = x(t) where q is any integer and the period is 4.

Calculate the complex Fourier Series coefficients and plot the amplitude and phase spectra.

Solution

A sketch of the signal is shown in Figure 7. Note, when a mathematical description of the signal is given, it is often a good idea to start by sketching the signal.

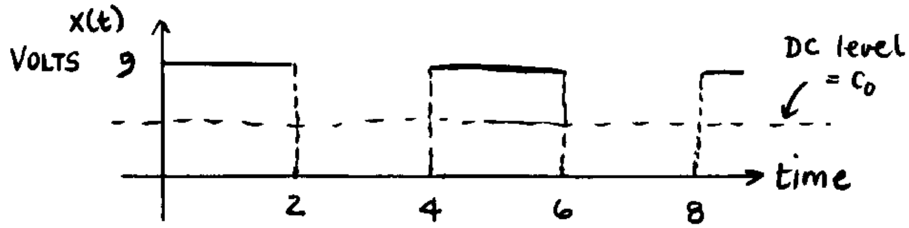


Figure 7: A sketch of the given signal, which is a square wave

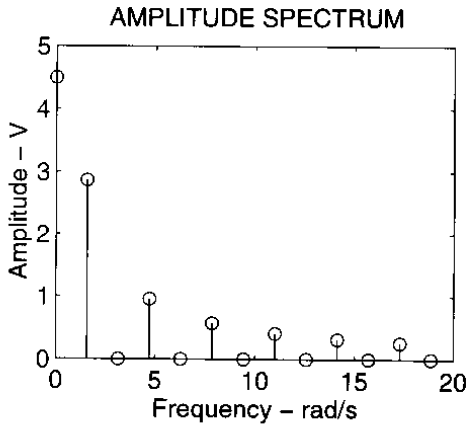
This signal repeats every 4 seconds, hence T=4 seconds, and ω_1, the fundamental frequency, = 2π/4 rad/s. The D.C. or zero frequency component is 4.5 Volts, from inspection. Therefore, c_0 = 4.5 Volts. The complex Fourier Series coefficients, for k≠0, are:

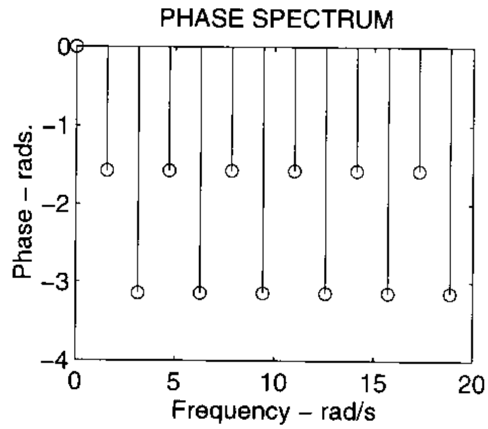
$$c_k = \frac{1}{T}\int_0^T x(t)\, e^{-j\frac{2\pi k}{T}}\, dt = 0.25\int_0^2 9\, e^{-j0.5\pi kt}\, dt$$

$$= \left| \frac{2.25}{-j0.5\pi k} e^{-j0.5\pi kt} \right|_0^2 = -4.5\frac{j}{\pi k}\ 1 - e^{-j\pi k}$$

$$= -4.5\frac{j}{\pi}\ 1 - \cos\pi k + j\sin\pi k\ = -4.5\frac{j}{\pi k}\ 1 - \cos\pi k$$

You might notice that whenever k is an even integer the coefficient is zero. When k is an odd – integer the coefficient is $-j9/\pi k$. The c_k are all purely imaginary for k positive and the imaginary part is negative. Hence the phase for all non-zero c_k for k>0 is –π/2 radians. For k even, the phase is indeterminate; when plotting by hand we usually set the phase to zero for this case. Shown below are the amplitude and phase spectra for this signal. Since c_0 is positive, the phase at $\omega = 0$ is 0. Had c_0 been negative the phase at $\omega = 0$ would have been –π.





Truncation and Convergence of Fourier Series

Where x(t) is continuous the Fourier series converges to the value of x(t) At points of discontinuity in the function x(t), such as those at $T_1, T, T + T_1$, etc., in the signal shown in Figure 5, the Fourier series converges to the average value of x(t) on either side of the discontinuity ($X_1/2$ in this case). This is a general rule. This is also illustrated in Figure 9, where the components of a square wave are shown along with partial sums of the components.

You might note that as more terms are added together the approximation over the flat, continuous part of the sine wave improves. However, the approximation is always poor close to the discontinuity. As the number of terms increases, the region over which the approximation is poor gets smaller, but the overshoot close to the discontinuity continues to get worse. This is called **Gibbs phenomenon** and is caused by approximating a discontinuous function with a finite series of continuous functions.

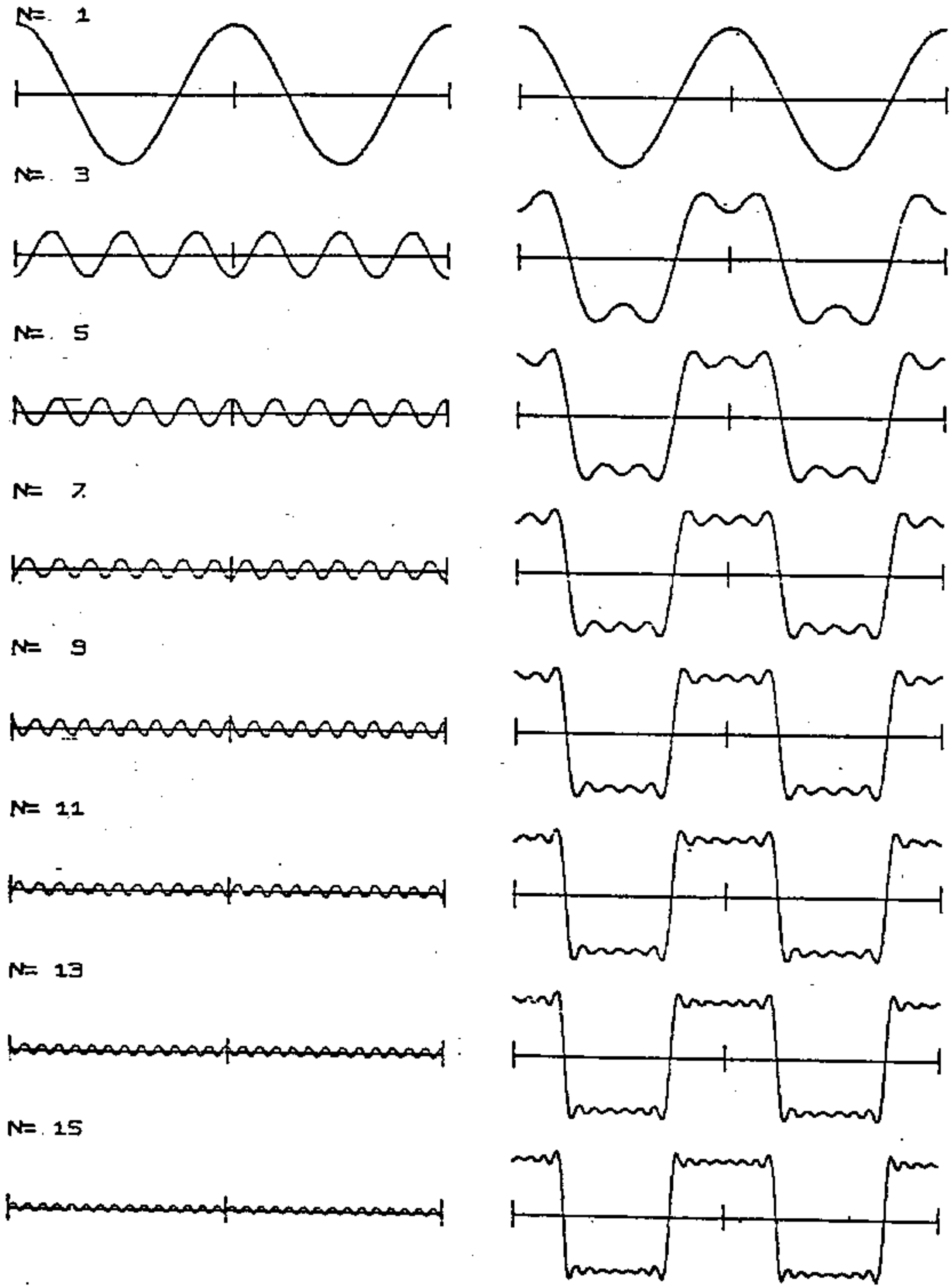


Figure 9: Left column: the individual components in the Fourier Series,
right column: the partial sum of terms in the Fourier series

Tricks to Simplify Calculation of the Fourier Coefficients

Evaluation of the Coefficients can often be simplified by noting the following:

1. Any D.C. component of the function x(t) appears only in the $A_0/2 = c_0$ term.
   Thus the A_k, B_k coefficients will be identical for the two functions shown below.

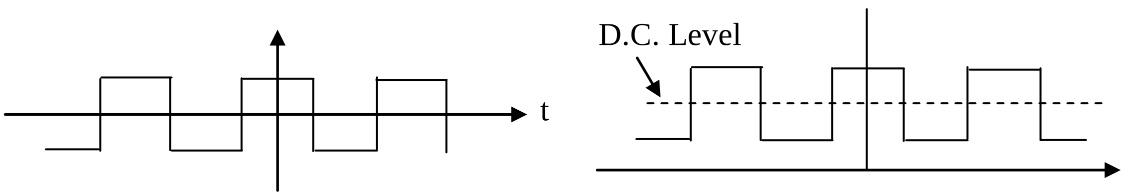


Figure 10: Two signals whose Fourier series are identical except for $A_0/2 = c_0$

2. $c_0 = A_0/2$ is the average value of the function x(t).
   Thus for the square wave on the left above $A_0/2 = 0$. For the square wave on the right $c_0 = A_0/2 =$ the D.C. level indicated.

3. The limits of integration can be changed.
   Although the integrals defining the Fourier coefficients were given in terms of the limits [0,T], the equations are valid for any interval of length T since we are dealing with periodic functions. Thus the limits of integration could be, for example, [-T/2, T/2]. As an example consider the function shown below.

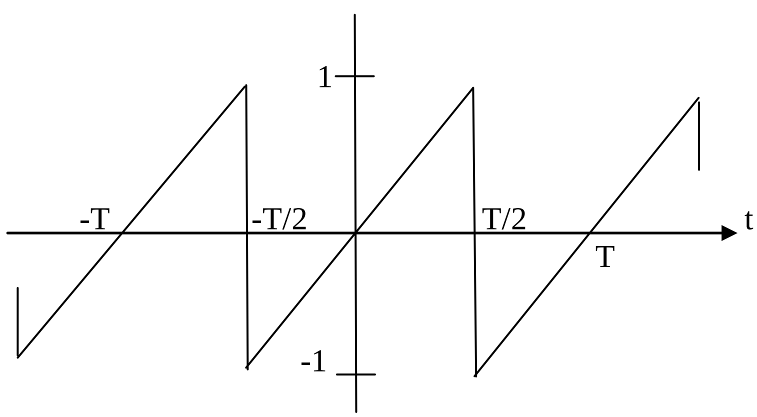


Figure 11: Saw-tooth wave

For the interval [0,T]

$$x(t) = \frac{2t}{T} \quad 0 \le t < T/2, \quad \text{and} \quad x(t) = \frac{2}{T}(t - T) \; \frac{T}{2} \le t < T$$

This approach requires that the integral for each coefficient be divided into two separate integrals. Consider instead the interval $-T/2, T/2$

$$x(t) = \frac{2}{T}t \qquad -\frac{T}{2} \le t < T/2$$

This form considerably reduces the amount of work required to evaluate the coefficients.

4. Absence of sine and cosine terms for even and odd functions, respectively.
   Even Functions, e.g., cosines: x(-t) = x(t), the function forms a mirror image about the t=0 axis.

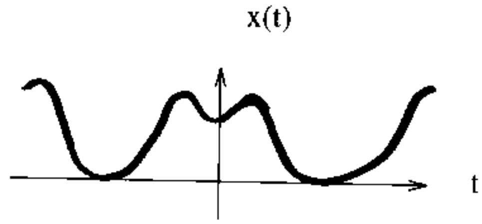


Figure 12(a): Even symmetry.

The Fourier series of this function would contain cosine terms only, i.e., the $B_k = 0$. If you are using the complex form this would mean that the coefficients would be purely real.

Odd Functions, e.g., sines: x(-t) = -x(t), there is symmetry about the origin .

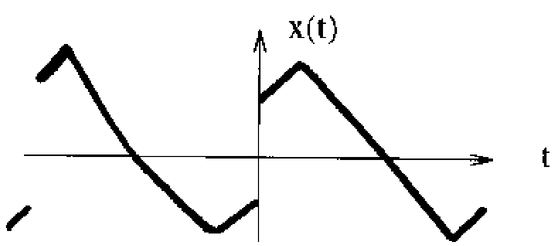
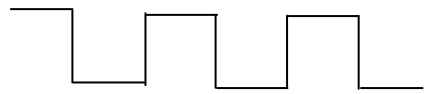


Figure 12(b): Odd symmetry.

The Fourier series of this function would contain sine terms only, i.e., the $A_k = 0$. If you are using the complex form this would mean that the coefficients would be purely imaginary.

In questions that require you to consider an arbitrary square wave, choose the alignment of the function so that all the $B_k = 0$ (case I, shown in Fig. 13), or all the $A_k = 0$, (case II, shown in Fig. 13). This will reduce the number of calculations, if you are using the sine and cosine form of the Fourier series.



Arbitrary square wave

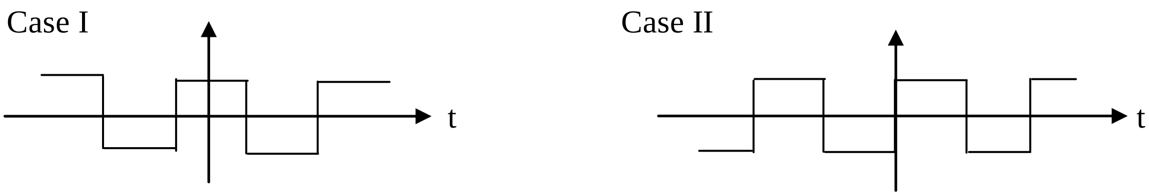


Figure 13: Aligning functions to be either even or odd

Plotted below is the amplitude spectrum for the Case II square wave shown in Figure 13. $A_0 = 0, A_k = 0$ (odd function) and $B_k = \frac{X_1}{k\pi}\left[1-(-1)^k\right]$. Note that the amplitude of the frequency components decreases as l/k. Thus one could truncate the series expansion after about 20 terms and have a reasonably good representation of the function (see Figure 9). In other words the signal has only a small amount of frequency content beyond $20\omega_1$. This means that an instrumentation system must have a cutoff frequency of at least 5 to 10 times $20\omega_1$, if the signal is to be transmitted with minimal distortion (see next section for the rationale behind 5 to 10 times the highest frequency). Note, however, that Gibb's effect will still occur near any discontinuities due to truncation.

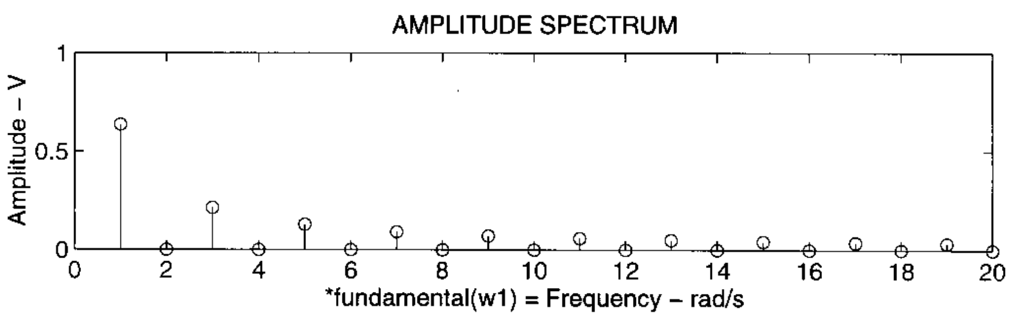


Figure 14: Line spectrum of a square wave function with odd symmetry

Examples of Fourier Series

l. <u>Pulse Wave of Period T (see Figure 5)</u>

$$x(t) = X_1 \quad 0 \le t \le T_1, \quad x(t) = 0 \quad T_1 < t < T$$

$$\frac{A_0}{2} = \frac{1}{T}\int_0^{T_1} x(t)\,dt = \frac{1}{T}\left[\int_0^{T_1} X_1\,dt + \int_{T_1}^{T} 0\,dt\right] = \frac{X_1 T_1}{T}$$

$$A_k = \frac{2}{T}\int_0^T x(t)\cos k\omega_1 t\, dt$$

$$= \frac{2}{T}\left[\int_0^{T_1} X_1 \cos k\omega_1 t\, dt + \int_{T_1}^T 0 \cdot \cos k\omega_1 t\, dt\right] = \frac{2X_1}{T}\int_0^{T_1}\cos k\omega_1 t\, dt$$

$$= \frac{2X_1}{T}\left[\frac{1}{k\omega_1}\sin k\omega_1 t\right]_0^{T_1} = \frac{2X_1}{T}\cdot\frac{T}{2\pi k}\sin\frac{2\pi k T_1}{T} = \frac{X_1}{k\pi}\sin\frac{2\pi k T_1}{T}$$

$$B_k = \frac{2}{T}\int_0^T x(t)\sin k\omega_1 t\, dt$$

$$= \frac{2}{T}\left[\int_0^{T_1} X_1 \sin k\omega_1 t\, dt + \int_{T_1}^T 0 \cdot \sin k\omega_1 t\, dt\right] = \frac{2X_1}{T}\int_0^{T_1}\sin k\omega_1 t\, dt$$

$$= \frac{2X_1}{T}\left[-\frac{1}{k\omega_1}\cos k\omega_1 t\right]_0^{T_1} = \frac{X_1}{T}\cdot\frac{T}{k\pi}\left[1 - \cos\frac{2\pi k T_1}{T}\right]$$

2. <u>Square Wave - Pulse Wave with $T = 2T_1$</u>

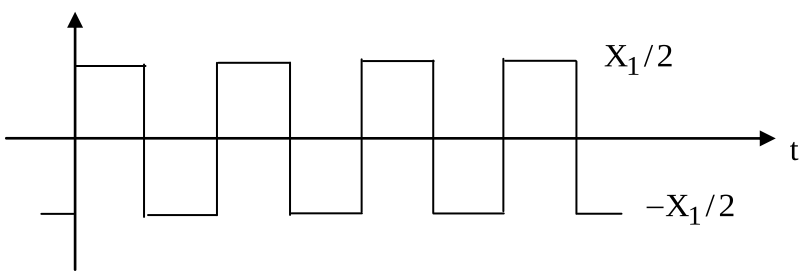


Figure 15: Square wave

This can be analyzed as a special case of the function in example 1 above, with $T_1/T = 1/2$.

$A_0 = 0$ since the average value of the function above is zero.

$$A_k = \frac{X_0}{k\pi}\sin\frac{2\pi k T_1}{T} = \frac{X_0}{k\pi}\sin \pi k = 0.$$

This was expected because the function is odd.

$$B_k = \frac{X_0}{k\pi}\left[1-\cos\frac{2\pi k T_1}{T}\right] = \frac{X_0}{k\pi}\;1-\cos\pi k\;\; = \frac{X_0}{k\pi}\left[1-(-1)^k\right]$$

3. <u>Triangular Wave</u>

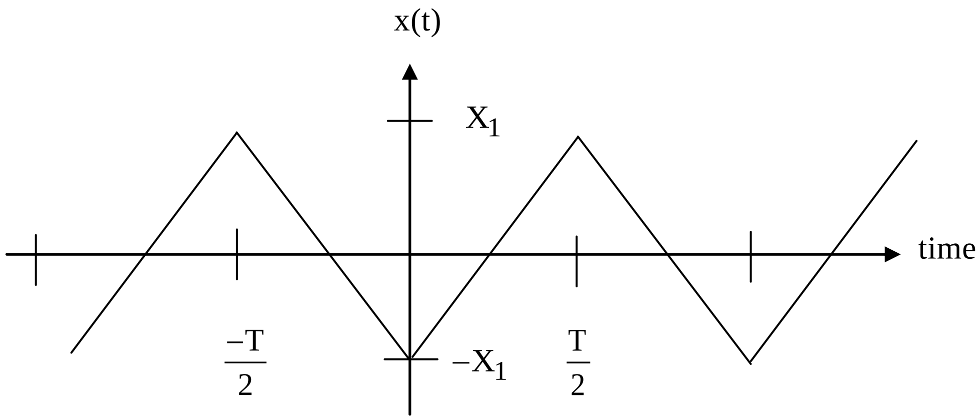


Figure 16: Triangle wave

$A_0 = 0$ since average value of function is zero
$B_k = 0$ since function is even

In order to evaluate A_k, we must integrate over one complete cycle of the original time function. We could, for example, integrate from t = 0 to t = T. But this choice is not as convenient because the mathematical description of x(t) is slightly more difficult to derive for $\frac{T}{2} \le t \le T$ than for $-\frac{T}{2} \le t \le 0$. Let us choose to integrate over the range $-\frac{T}{2} \le t \le +\frac{T}{2}$. To verify that the $B_k = 0$, both integrals will be evaluated.

$$A_k = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\cos k\omega_1 t\,dt\,, \quad B_k = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\sin k\omega_1 t\,dt$$

Next we need to write expressions for x(t) over the entire integration range. For $-T/2 \le t \le 0$: x(t) is linear function, therefore $x(t) = m_1 t + b_1$. Use $x(-T/2) = X_1$ and $x(0) = -X_1$, to give

$$m_1 = -\frac{2X_1}{T/2} = -\frac{4X_1}{T};\ \text{and}\ b_1 = -X_1$$

For $0 \le t \le +T/2$: x(t) is again a linear function: $x(t) = m_2 t + b_2$. Use $x(0) = X_1$ and $x(T/2) = X_1$, to give

$$m_2 = -m_1 = +\frac{4X_1}{T} \quad \text{and} \quad b_2 = b_1 = -X_1$$

So we have:

$$A_k = \frac{2}{T}\left[\int_{-T/2}^{0}\left(-\frac{4X_1}{T}t - X_1\right)\cos k\omega_1 t\,dt + \int_{0}^{+T/2}\left(+\frac{4X_1}{T}t - X_1\right)\cos k\omega_1 t\,dt\right]$$

$$B_k = \frac{2}{T}\left[\int_{-T/2}^{0}\left(-\frac{4X_1}{T}t - X_1\right)\sin k\omega_1 t\,dt + \int_{0}^{+T/2}\left(+\frac{4X_1}{T}t - X_1\right)\sin k\omega_1 t\,dt\right]$$

We now proceed to integrate the above equations either by the use of handbook integral tables or by known analytical methods. The results are:

$$A_k = \frac{4X_1}{k^2\pi^2}(\cos k\pi - 1) = \frac{-8X_1}{k^2\pi^2}; \;\; n = 1,3,5,7,\ldots \qquad A_k = 0; \quad k = 2,4,6,\ldots$$

$B_k = 0$ for all k.

Note that, in this case the A_k will decrease at the rate of $1/k^2$ instead of 1/k as in the preceding square wave case.

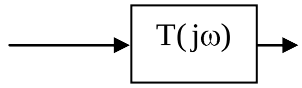
We could have derived the result from the previous example by integrating the Fourier Series result for the square wave. Integrating the signal shown in Figure 15 would result in a triangular wave of amplitude X_1 T/8. Hence multiplying the Fourier coefficients of the square wave by 8/T and integrating the kth term in the Fourier series should give the same result. You should check that it does.

When you integrate, the sine terms become cosines and vice versa. Integrating Fourier series is fine, but be careful differentiating them, because you may have infinite derivatives at discontinuities in the signal and a Fourier series of that signal may not exist.

Passing Periodic Signals Through Systems

We are now in a position to analyze the effect of an instrumentation system on a general periodic input signal. Consider such a system represented by an overall transfer function $T(j\omega)$.

Input Signal



Output Signal

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} M_k[\cos(k\omega_1 t - \theta_k)]$$

y(t)

The output signal we can generate from the frequency response function and using its relationship to the steady state response of the system to a sinusoidal input.

$$y(t) = \frac{A_0}{2} T(j0) + \sum_{k=1}^{\infty} M_k \left|T(j\omega_k)\right| \cos\ \omega_k t - \theta_k + \angle T(j\omega_k) \qquad (21)$$

The *amplitude spectrum of the output* is defined by:

$$\left|\frac{A_0}{2} T(j0)\right| \text{ for } \omega = 0 \text{ rad/s}$$

$$\text{and} \quad M_k^{\text{output}} = M_k T(j\omega_k) \quad \text{for} \quad \omega = \omega_k = k\omega_1 \text{ rad/s} \qquad (22a)$$

The *phase spectrum of the output* is defined by:

0 if the D.C. term is positive, and π if the D.C. term is negative, when $\omega = 0$ rad/s.

$$\text{and} \quad \theta_k^{\text{output}} = \theta_k - \angle T(j\omega_k) \text{ for } \omega = \omega_k = k\omega_1 \text{ rad/s.} \qquad (22b)$$

The requirement for the output to look like the input signal is:

$$y(t) = K\,x(t - t_0)\,, \qquad (23)$$

i.e., all components in the signal are scaled by the same amount and all have the same time delay. So, in the frequency region where all the major components of the input signal lie, we would like

$$\left|T(j\omega_k)\right| \approx \text{constant} \qquad (24a)$$

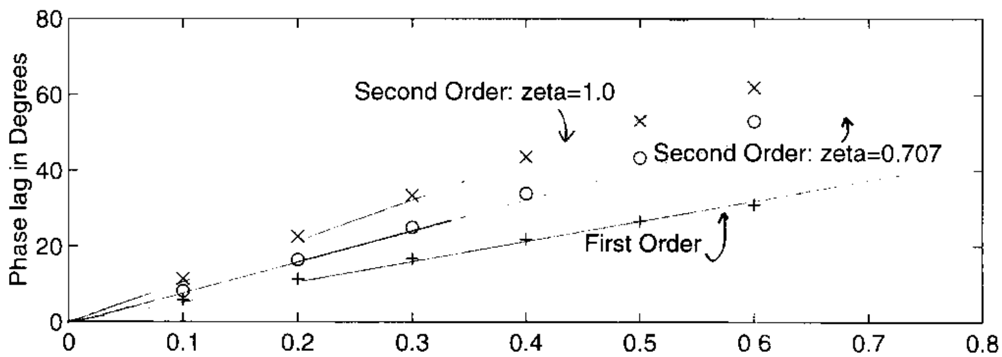
$$\text{and} \quad \angle T(j\omega_k) \approx -\omega_k t_0\,, \qquad (24b)$$

for some time delay t_0. A time delay in the signal results in a phase that is a linear function of frequency, because

$$\cos\ \omega_k(t - t_0)\ = \cos\ \omega_k t - \omega_k t_0$$

and the phase of this is $-\omega_k t_0$.

Many of the measurement systems we use can be modeled as first or second order systems. So over what frequency range in these systems do these constant gain and linear phase conditions hold approximately? The answer depends on what error you are willing to tolerate. Rule of thumb estimates may be 0 to one-seventh of the cut-off frequency ($1/\tau$) for a first order system, and 0 to one-tenth the natural frequency (ω_n) for a second order system with a damping ratio less than 1. In Figure 17 is shown the actual phase lag versus frequency for a first order and two second order systems.



Nondimensional frequency, $\tau\omega$ or $\dfrac{\omega}{\omega_k}$

Figure 17: Phase behavior of first and second order systems.

SPECTRA OF NON-PERIODIC SIGNALS, THE FOURIER TRANSFORM

Consider the even periodic function shown below

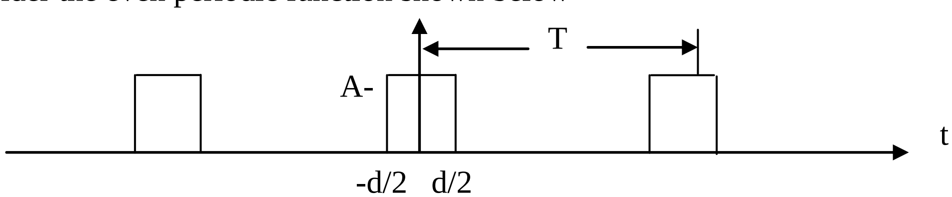


Figure 18: x(t), a rectangular pulse train

Note that this is an even function, and the Fourier series expansion is:

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos k\,\omega_1 t, \quad \text{where } \frac{A_0}{2} = \frac{Ad}{T} \text{ and } A_k = \frac{2Ad}{T}\frac{\sin(k\pi d/T)}{(k\pi d/T)}.$$

Thus $\dfrac{A_k}{A_0} = \dfrac{\sin(k\pi d/T)}{(k\pi d/T)} = \dfrac{\sin(k\omega_1 d/2)}{(k\omega_1 d/2)}$, which is plotted in Figure 19.

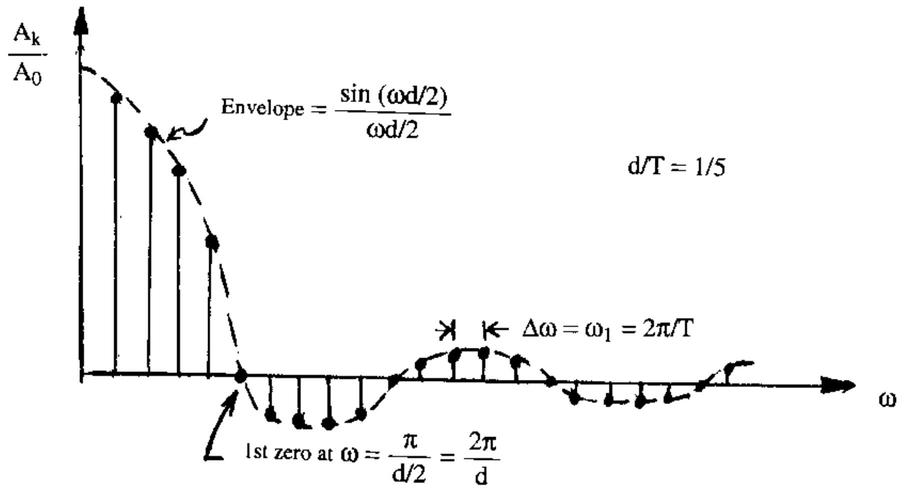


Figure 19: Spectrum for the signal shown in Figure 18

Note the following things about the line spectrum:

1. The spectral components are separated in frequency by the fundamental frequency $\omega_1 = 2\pi/T$. As T increases the envelope remains the same, but the spectral components become more closely spaced.

2. The first zero in the envelope occurs at $\omega = 2\pi/d$. Therefore the "width" of the spectrum is inversely related to the width of individual pulses in the signal.

3. Over certain intervals the Fourier coefficients are negative. This corresponds to a phase shift of π for these components.

Now consider what could happen if the period T were increased without limit. The signal would become a single square pulse at the origin and its spectrum would contain <u>all</u> frequencies (refer to note 1 above). Thus the spectrum of a non-period function is <u>continuous</u> rather than discrete. From note 2 we observe that a pulse which is narrow in time will have a frequency spectrum which is broad and vice versa. This is illustrated below.

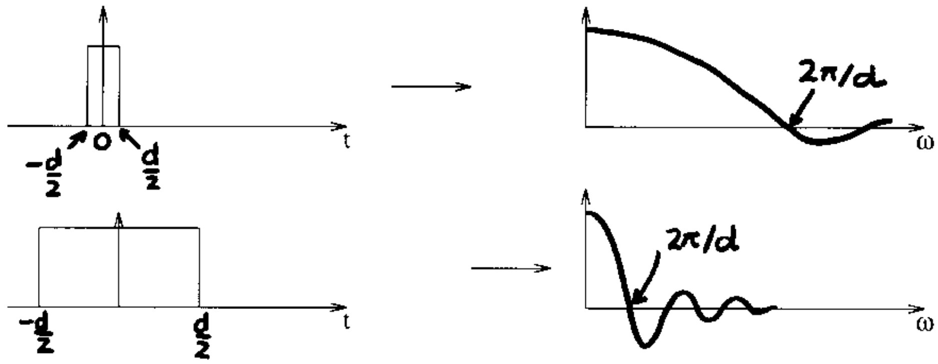


Figure 20: Effect of pulse width on spectrum

We see that for narrow pulses to be transmitted with minimal distortion the transmitting device must have a large bandwidth. This arises, for example, in digital data processing systems where the information is in the form of narrow voltage pulses.

Our development of the spectral behavior of non-periodic signals from the standpoint of Fourier series has been heuristic in nature. The exact mathematical treatment requires the use of the Fourier transform, as described below.

The Fourier Transform

Starting from equations (17) and (18), the complex form of the Fourier series, note that 1/T equals the spacing between frequency components (in Hertz), which we have denoted by $f_1 = \omega_1/2\pi$. Rewrite the equations for x(t) and c_k as

$$x(t) = \sum_{kf_1=-\infty}^{\infty} \frac{c_k}{f_1} e^{j2\pi k f_1 t} f_1 \tag{25a}$$

and

$$X(kf_1) \triangleq \frac{c_k}{f_1} = \int_{-T/2}^{T/2} x(t) e^{-j2\pi k f_1 t}\, dt \tag{25b}$$

Now we invoke a limiting process by letting $T \to \infty$. Thus $f_1 \to 0$, i.e., the frequency separation between adjacent spectral components approaches zero and $kf_1 \to f$, i.e., a transition to a continuous variable has been made. Furthermore $f_1 \to df$ and $\frac{c_k}{f_1} \to X(f)$, where X(f) is a continous function of frequency. The expressions for x(t) and X(f) are then

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi t} df \qquad (26a)$$

and

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt \qquad (26b)$$

These last two equations represent a Fourier transform pair where X(f) is the Fourier transform of x(t). It is the (continuous) spectral representation of x(t). Conversely, x(t) is the inverse Fourier transform of X(f). The Fourier transform can be found for any signal for which the integral of $|x(t)|$ from $-\infty$ to ∞ exists and for which any discontinuities are finite. (These are sufficient conditions. There are signals which violate at least one of these conditions and still have a Fourier transform.)

Consider the following example of a periodic function of rectangular pulses.

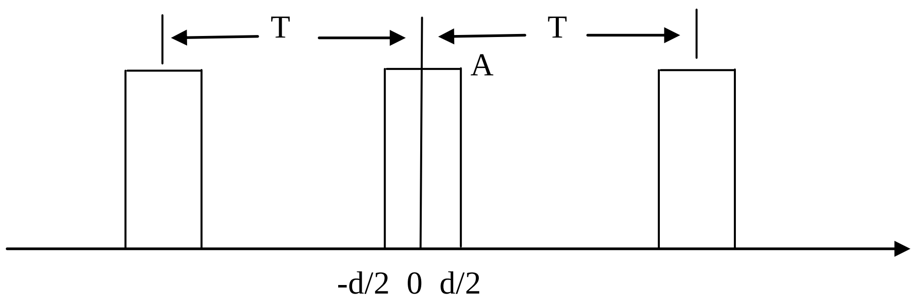


Figure 21: A rectangular pulse train.

If we let $T \to \infty$ we are left with only the central pulse which is the non-periodic signal x(t). Then the Fourier transform is

$$X(f) = \int_{-d/2}^{d/2} A\, e^{-j2\pi ft} dt = \left| \frac{-A\, e^{-j2\pi ft}}{j2\pi f} \right|_{-d/2}^{d/2}$$

$$X(f) = \left[\frac{-A}{j2\pi f}\right] \cos 2\pi f(d/2) - j \sin 2\pi f(d/2) \ - \left[\frac{-A}{j2\pi f}\right] \cos 2\pi f(-d/2) - j \sin 2\pi f(-d/2)$$

Recall that $\cos(\theta) = \cos(-\theta)$, and $\sin(\theta) = -\sin(-\theta)$. Thus

$$X(f) = \frac{A(2\sin \pi df)}{2\pi f} = Ad\frac{\sin(\pi df)}{\pi df} = Ad\, \mathrm{sinc}(\pi df)$$

This is shown in Figure 22. Note that in the derivation of the Fourier transform we took the Fourier series coefficients and divided by f_1. We then let the period go to infinity and the line spectrum became a continuous spectrum. The units of the magnitude of the Fourier Transform are therefore, x(t)'s units divided by frequency. So if we measured the signal in volts, $|X(f)|$ would have units volts/Hertz.

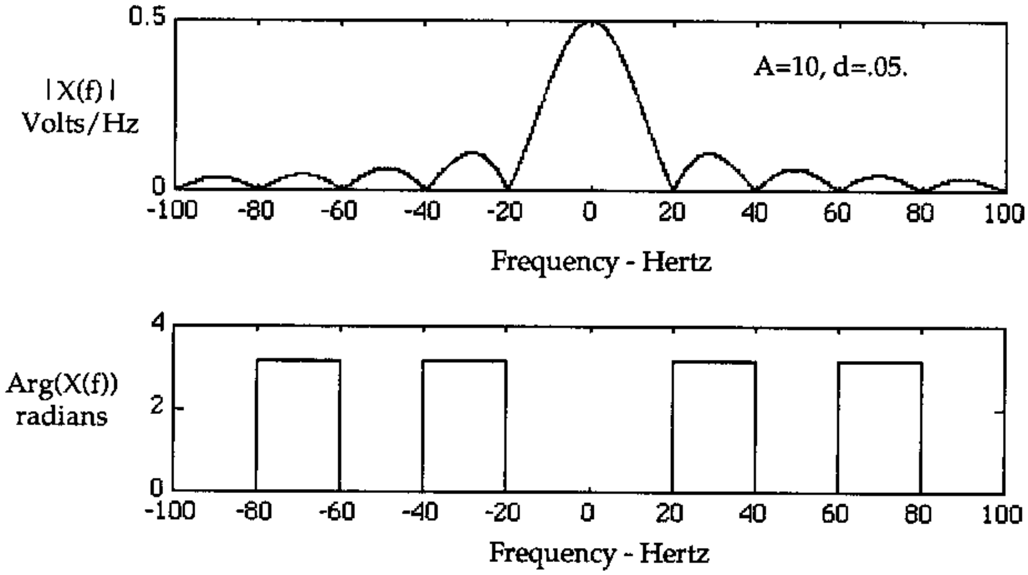


Figure 22: The Fourier transform of a Single Rectangular Pulse of Height A and Width d.
(Negative values Correspond to Phase Shifts of π)

The Fourier transform is a powerful spectrum analysis tool with wide application. To perform these Fourier transforms on signals coming out of measurement systems, we usually sample the signals and store the samples on a computer. We then analyze these sequences of samples with the Discrete Fourier Transform (DFT). The DFT is related to the Fourier transform described above, but the sampling and numerical evaluation will affect the results. However, if care is taken you can use the DFT to estimate the Fourier series coefficients of a periodic signal, and the Fourier transform of a non-periodic signal. In the next section we show how this is done using the FFT program in MATLAB. The Fast Fourier Transform (FFT) is a Discrete Fourier Transform that is computationally efficient.

THE DISCRETE FOURIER TRANSFORM (DFT)

Calculating the Fourier Series Coefficients by using the DFT

To calculate the Fourier series coefficients on a computer we have to sample the time history (beware of aliasing) and perform the integration numerically. Suppose we use a simple rectangular integration, as shown in the figure below for the real part of the integral. Note we are using the notation c_k to represent the Fourier series coefficients, k=......-2,-1,0,1,2,3..... We will use X_k to denote the result of the discrete Fourier transform, note that this is slightly different to the notation used in earlier sections of this chapter. In addition we will assume that the integration time is T seconds where T is a whole number of periods of the signal.

$$c_k = \frac{1}{T} \int_{\text{Mperiods}} x(t) e^{-j2\pi\frac{kt}{T}} dt = \frac{1}{T} \int x(t) \cos\left(2\pi\frac{kt}{T}\right) dt - \frac{j}{T} \int x(t) \sin\left(2\pi\frac{k}{T}t\right) dt$$

On a digital computer we will not know x(t) as a continuous function of time (t). We will have x(t) at times $t = n\Delta$ where n=0,1,2,3..... Furthermore we will assume that we have sampled in such a way that $T = N\Delta$ represents a whole number of periods. In Figure 23, N = 20 and NΔ is exactly the time span of three periods of the signal. We now do the integration numerically by using rectangular integration. This is illustrated graphically in Figure 23, for some value of k. x(t) $\sin(2\pi kt/T)$, is the real part of the integrand, and the sum of the shaded areas is an approximation to the real part of Tc_k.

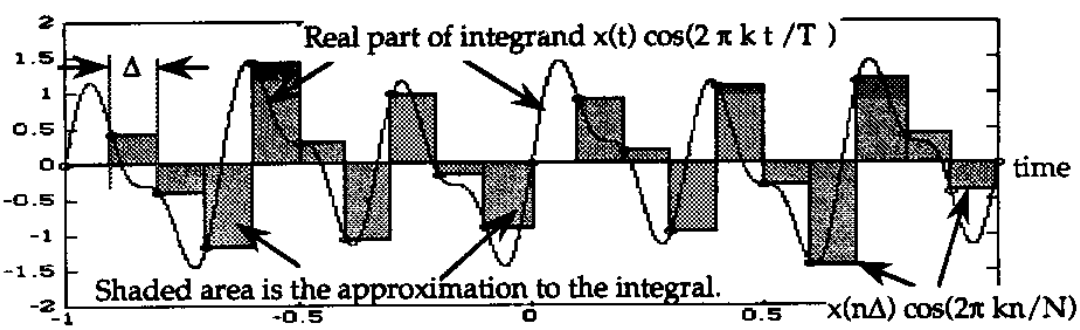


Figure 23: Rectangular integration to approximate the real part of c_k

$$c_k \approx \frac{1}{N\Delta} \sum_{n=0}^{N-1} x(n\Delta)\, e^{-j2\pi\frac{nk}{T}\Delta} \approx \frac{1}{N} \sum_{n=0}^{N-1} x(n\Delta)\, e^{-j2\pi\frac{nk}{N}}$$

The DFT is often defined as $X_k = Nc_k = \sum_{n=0}^{N-1} x(n\Delta)\, e^{-j\frac{nk}{N}2\pi}$.

You can use a discrete Fourier transform (DFT) to calculate the Fourier series coefficients c_k provided:

(1) you don't alias when sampling, and

(2) N samples represent an integer number of periods of the signal, which in turn means that your sample rate must be directly linked to the fundamental frequency of the periodic signal.

You will usually use an FFT algorithm to calculate the DFT, and sometimes this algorithm will require N to be a power of 2, $N = 2^q$ which further restricts your choice of sample rate.

So basically

$$\text{fundamental frequency} = \frac{1}{T_p} = \frac{f_s}{2^q} \cdot M$$

where M = some integer and $f_s > 2\, f_{max}$, where f_{max} is the highest frequency in the signal.

$$\therefore\ f_s = \frac{2^q}{T_p \cdot M} > 2\, f_{max}$$

A further restriction comes from the analog to digital converter that you use when sampling your signal, which may not always allow you to sample at the rate you wish. Below is a listing of a MATLAB .m file for calculating the Fourier series coefficients of a signal of period .1 second. Ten periods are used in the transform, which is why the tenth coefficient is the first non-zero one, as illustrated in Figure 24.

```
% four.m program to calculate the Fourier Series coefficients
% of a signal consisting of two sines
%
% First calculate signal
clg
t=(0:999)/1000;
a=9*cos(2*pi*10*t) + 5*cos(2*pi*200*t);
%
% Now FFT result - MATLAB does not require a power of two
%
y=fft(a)/1000;
%
% Calculate the first 500 frequencies
f=(0:499);
%
%Plot the real and imaginary parts of the first 500 coefficients
subplot (211);
plot(f,real(y:500)),'+')
xlabel (‘REAL PART CK’ )
```

```
subplot(212)
plot(f,imag(y(1:500)),'+')
xlabel('FREQUENCY – HERTZ')
ylabel ('IMAG PART CK')
```

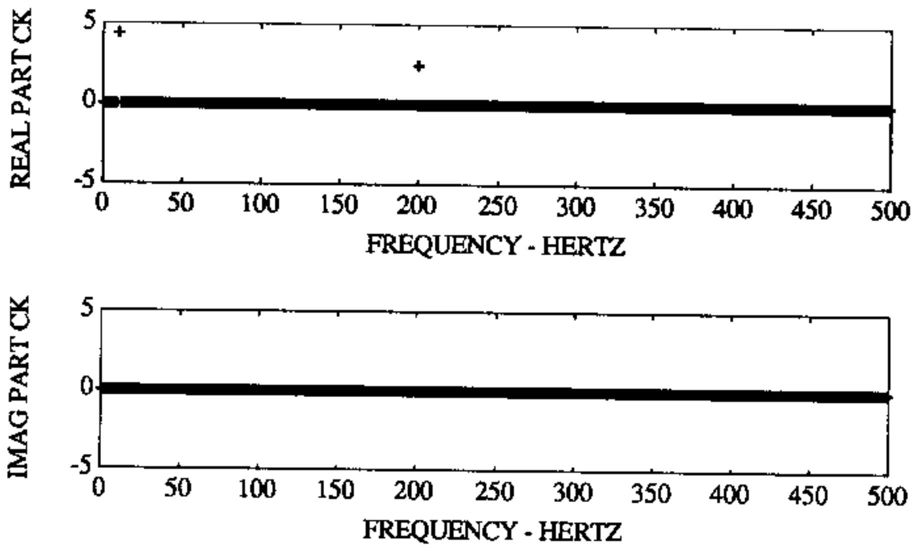


Figure 24: The real and imaginary parts of the Fourier coefficients.
Note $c_k = \frac{1}{2}(a_k - j\, b_k), a_k$ and b_k are real.

Calculating the Fourier Transform of a Signal using the DFT

To use the DFT to calculate the Fourier Transform again use rectangular integration so:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \approx \sum_{n=-\infty}^{\infty} x(n\Delta)\, e^{-j2\pi fn\Delta} \Delta\,.$$

You can only evaluate this at a finite number of frequencies $f_k = \frac{kf_s}{N} = \frac{k}{N\Delta}$.

$$X(f_k) \approx \Delta \sum_{n=-\infty}^{\infty} x(n\Delta)\, e^{-j2\pi \frac{kn}{N}}\,.$$

Similarly we cannot do infinite sums so we take the first N points: n=0,1,2 ...... N-1.

$$X(f_k) \approx \Delta \sum_{n=-0}^{N-1} x(n\Delta)\, e^{-j2\pi \frac{kn}{N}} = \Delta\, \text{DFT}\,\{x(n\Delta)\} = \Delta\, X_k\,.$$

Here you have truncated the signal and you may have aliased, both will affect the results.

SPECTRAL CONTENT OF RANDOM SIGNALS

A detailed theory of random signals is beyond the scope of these notes, but it is possible to make some general observations based on our discussion to this point. A rough idea of the spectral content of a random signal may be obtained by simple observation. Consider, for example, the signal shown below. $\phi_x(f)$ is the power spectral density of x(t) and is defined below.

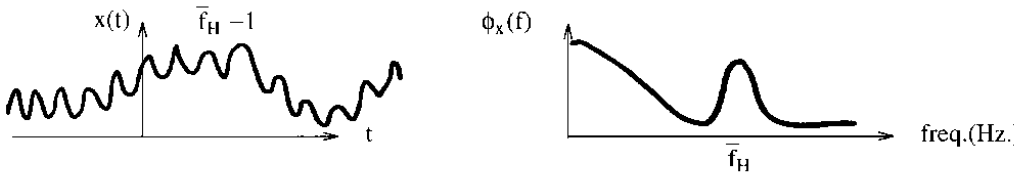


Figure 25: Power Spectral Density (PSD) of a random signal

The signal, x(t), has a relatively slow mean variation with time, indicating low frequencies present in the signal, and also some very low amplitude fluctuations of higher frequency superimposed. We would therefore expect the spectrum to look something like that shown at the right. The bump at $\bar{f}_H$ is centered at the mean frequency of the small high frequency fluctuations. If the signal was transmitted through a system of low bandwidth these would be removed. A rough estimate of the width of the primary low frequency peak could be obtained by estimating the mean period of the low frequency variations.

The power spectral density of a signal shows the distribution of the average signal power across frequency. This would suggest that the units of the power spectral density should be Watts/Hertz. However, while the term *power* is used the units are actually (signal units)^2 per Hertz. Sometimes the output is converted to Watts by considering the signal to be the voltage across a 1 Ω resistor. The power dissipated by the resistor would then be $\{V^2/R\} = V^2$ Watts.

Average signal power can be calculated from the time history by using:

$$\text{Signal Power} = \text{average}\left[x^2(t)\right] = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)\, dt$$

It can be shown that this is equal to:

$$\int_0^{+\infty} \phi_x(f)\,df$$

where ϕ_x denotes the power spectral density of x(t). If you are interested in how much signal power is concentrated in a band from f_1 to f_2 Hertz, then you can calculate this from the power spectral density by integrating it between these frequencies:

$$\text{power in band } f_1 \le f < f_2 = \int_{f_1}^{f_2} \phi_x(f)\,df$$

This is illustrated in Figure 26.

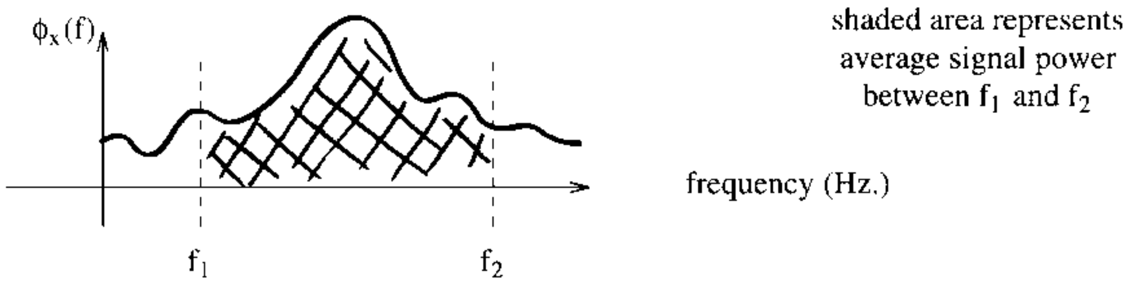


Figure 26: Illustration of the calculation of signal power in a frequency band.

Relationship Between the Fourier Transform and the Power Spectral Density of a Signal

Suppose we have a time history that is random. If we take T seconds of the signal and Fourier Transform this segment, calculate the square of the magnitude and divide by T, we will generate a two sided version of $\phi_x(f)$. Remember that when we Fourier Transform we get values at negative frequencies as well as at positive frequencies. If we add the positive and negative frequency contributions together we will get $\hat{\phi}_x(f)$. So our estimate is:

$$\hat{\phi}_x(f) = \frac{2\left|X_T(j2\pi f)\right|^2}{T} \quad 0 < f < \infty$$

Note that we use the ^ notation to denote an estimate.

Now if we took the next T seconds of the signal and repeated the calculation we would get a different answer. This is because, if x(t) is random then $X_T(j2\pi f)$, the Fourier transform of T seconds of x(t), is also random. So to estimate $\phi_x(f)$ we take many segments of the signal, and average the $\hat{\phi}(f)$'s calculated from each segment. The result is still an estimate of $\phi_x(f)$.

To improve this estimate we must make T, the signal segment length, large and we must take many segments and average. The standard deviation of the average is the standard deviation of $\hat{\phi}_X(t)$ divided by $N^{1/2}$. This is just like our calculations of the standard deviation of the estimated mean in the statistics chapter.

The true power spectral density, $\phi_X(f)$, is defined to be:

$$\phi_X(f) = \lim_{T\to\infty} \mathbf{E}\left[\frac{2\left|X_T(j2\pi f)\right|^2}{T}\right]$$

where **E** [.] denotes *average value of*.

Now lets look at the units. The units of a Fourier Transform magnitude are:

{ signal units }/Hertz.

So squaring this and dividing by T (seconds) we get the units:

{ signal units }^{2}/(Hertz^2 seconds).

Since Hertz has units of seconds^{-1} then $\phi_X(f)$ has the units:

{ signal units }^{2} /Hertz.

Example

Suppose that a random signal is a white noise signal. This means that all frequencies are present in the signal and its power spectral density will be a straight line as shown in Figure 27(a) below. The level of the power spectral density is given as 25 Watts/Hertz. This signal is passed though a low-pass filter whose frequency response magnitude is shown in Figure 27(b). Calculate the total signal power of the signal coming out of the low-pass filter.

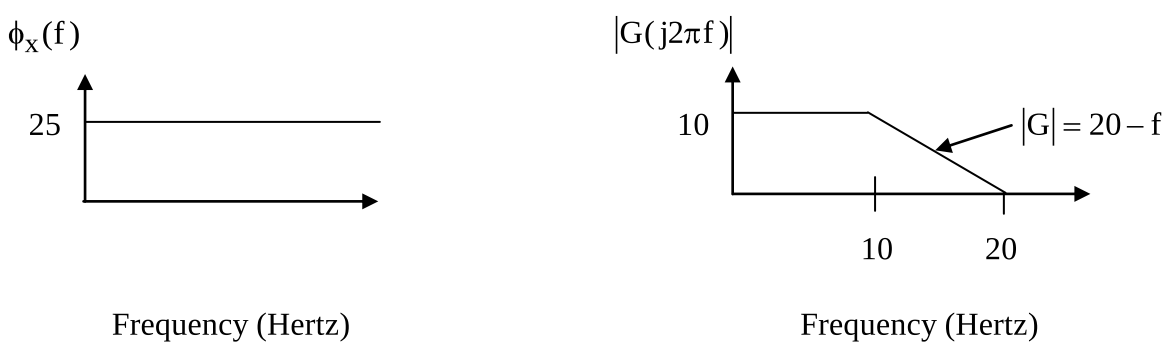


Figure 27: (a) PSD of white noise. (b) Frequency response magnitude of the low-pass filter.

Solution
Since we know that the Fourier Transform of the output of a filter is $G(j2\pi f)\,X(j2\pi f)$ where G is the frequency response of the filter and X is the Fourier Transform of the input, then we might expect, as is indeed the case, that the power spectral density of the output of the filter is:

$$\phi_y(f) = |G(j2\pi f|^2\,\phi_x(f)$$

The power spectral density of the response is shown in Figure 28.

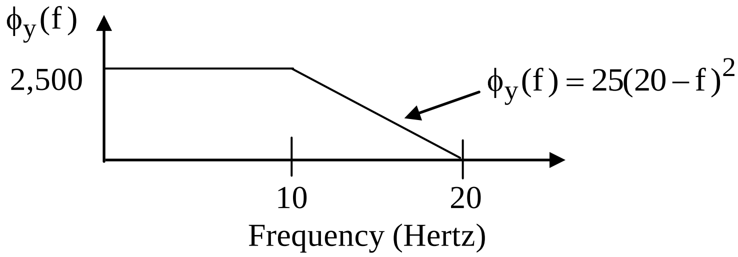


Figure 28: Power Spectral Density of filter output.

The total power of the filter output signal is the integration of this function from 0 to ∞.

$$\text{Signal Power} = \int_0^\infty |G(j2\pi f|^2\,\phi_x(f)\,df = \int_0^{10} 10^2\,25\,df + \int_{10}^{20} 20-f\ ^2\ 25\,df$$

$$= 25{,}000 + 1{,}250 \text{ watts.}$$

References

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