CHAPTER 7

FILTERS, LOADING AND OP-AMPS

INTRODUCTION

Sometimes we make measurements and what is measured is a combination of what we wished to measure and noise. This noise could be caused by the electronic circuitry, by external factors that affect the measurement system, or by external factors that affect the variable we are trying to measure. If this noise is in a different frequency range to the signal we can filter it out. A full discussion of noise, its sources and how its effects may be reduced are given in Chapter 11 of these course notes. This is one example of the use of filters.

However, any system will have a frequency response function associated with it and will amplify some frequency components and attenuate others. So in some sense all systems are filters. A thermocouple behaves as a first order system. If we plot its frequency response we will see that the magnitude becomes very small at high frequencies. So this is an example of a system that removes high frequencies from the signal. Very fast temperature fluctuations may not be measured with a thermocouple. We say that the thermocouple behaves like a *low pass* filter; it only allows low frequency components in signals to pass through. Some systems behave like *high pass* filters, they attenuate low frequencies and pass high frequencies. Some accelerometers made of quartz crystal have a frequency response that is small at very low frequencies, is flat for a range of frequencies, peaks in the region of the natural frequency of the crystal and then becomes very small at high frequencies. This type of system is a second order system combined with a high pass filter.

Band pass filters can be constructed by combining a low pass filter in series with a high pass filter as shown in Figure 1. These are often used in instrumentation to filter out low and high frequency noise, and also as part of a demodulation instrument to extract one channel of data. You use a band pass filter when you tune into a radio station.

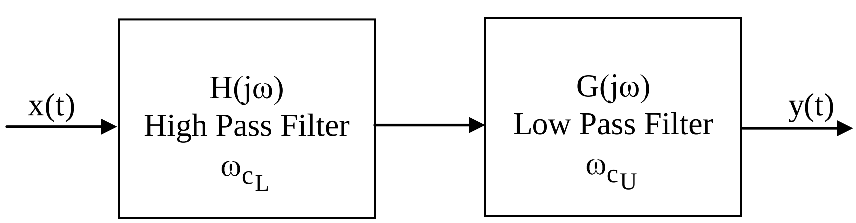

Figure 1: High and low pass filters combined to produce a band pass filter

It would be good if we could say that the frequency response function of the band pass filter was the product of the frequency response functions of the high pass and low pass filters.

However, this, in general, is not the case. This is because when systems are joined together they interact with one another. The frequency response function of the band pass filter is:

$$G(j\omega) = H_1(j\omega)\, H_2(j\omega)\, L_{12}(j\omega) \qquad (1)$$

where $L_{12}(j\omega)$ is the term that arises because of the interaction. We will refer to this as the *loading* term. Every time we add another subsystem in series we need to include another loading term. So, if in our measurement system we had three components plugged together in series, e.g., a thermocouple, an amplifier and a filter, and their frequency response functions were: $H_1(j\omega), H_2(j\omega)$ and $H_3(j\omega)$, respectively, the frequency response function of the combined system would be:

$$H_1(j\omega)\, H_2(j\omega)\, H_3(j\omega)\, L_{12}(j\omega)\, L_{23}(j\omega) \qquad (2)$$

We will describe how to determine $L_{12}(j\omega)$ and $L_{23}(j\omega)$ later in this chapter.

If the system is designed well the loading terms will be approximately equal to 1. In order to reduce the effects of loading it will be shown that we need to design systems with high input impedances and low output impedances. To help us do this we use operational amplifiers. The last part of this chapter is a brief description of how operational amplifiers work and how we use them to reduce the loading effects mentioned above.

PASSIVE FILTERS

Low Pass Filters: filters that attenuate high frequencies.

The simplest low pass filter is a first order system with a frequency response function:

$$G(j\omega) = \frac{K}{(1+\tau j\omega)} \qquad (3)$$

where K is the sensitivity (static) or *gain* of the system and τ is the system time constant. The cut-off frequency, ω_c, of the filter is where the magnitude of the frequency response function becomes $K/\sqrt{2}$, and K is the magnitude of the frequency response function when $\omega = 0$, i.e., at DC. If we plot the magnitude of the frequency response function in decibels, ω_c is where the magnitude has dropped by 3 dB from its value at $\omega = 0$. (You can see this from noting that $20\log_{10}(1/\sqrt{2}) = -3$ dB.) The magnitude and phase of the frequency response of this simple low pass filter are,

$$|G(j\omega)| = \frac{|K|}{(1+\tau^2\omega^2)^{1/2}}\ ,\ \arg(G(j\omega)) = -\tan^{-1}(\tau\omega) \text{ if } K \ge 0 \qquad (4)$$

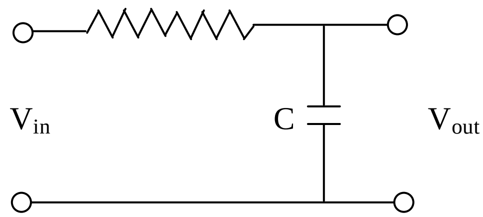

Figure 2: Simple RC circuit, low pass filter

In Figure 2 is shown a simple RC circuit which behaves like a low pass filter. By using, e.g., Kirchoff's laws and assuming that no current flows to whatever this circuit is connected to, it is straightforward to show that:

$$G(j\omega) = \frac{1}{(1 + jRC\omega)} \tag{5}$$

Note that we used the complex form of the impedances (because its easier) and Kirchoff's laws to get:

$$V_{out} = \frac{I}{Cj\omega}, \quad V_{in} = I\left(R + \frac{1}{Cj\omega}\right),$$

then eliminated the current, I, to come up with the result in (5). The magnitude and phase of $G(j\omega)$ are plotted in Figure 3. The time constant, $\tau = RC$ and the static sensitivity, K, (found here by setting $\omega = 0$) is 1.

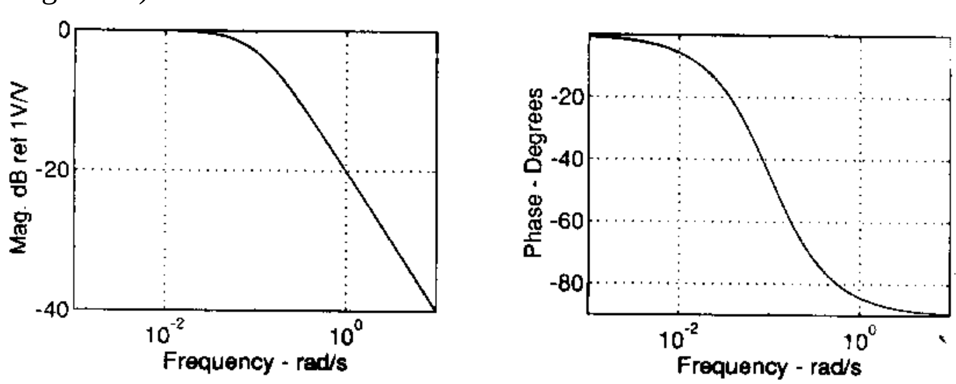

Figure 3: Magnitude and phase of the frequency response of a simple, first order, low pass filter

At high frequencies, the $j\omega RC$ in the denominator dominates the frequency response, so the response at 10ω is approximately 0.1 times the response at ω, giving a roll-off of 20 dB/decade, since $20\log_{10} 0.1 = -20\,\text{dB}$. If we had a frequency response function with a higher order polynomial in ω in the denominator we could have faster roll-offs. Suppose the denominator is a sixth order polynomial in ω and the top is a constant. At high frequencies, the frequency response magnitude is proportional to ω^{-6}. This means that at high frequencies the response at 10ω is approximately 10^{-6} times the response at ω. Since, $20\log_{10}10^{-6} = -120\,\text{dB}$, the roll off is 120 dB/decade. The cut-off frequency is still defined as the frequency at which the gain has dropped 3 dB from its initial value at $\omega = 0$ rad/s.

In Figure 4 is shown the frequency response of the sixth order low pass filter. Note that the phase also drops a lot more when the filter is higher order, instead of 90° it has dropped $6 \times 90^\circ = 540^\circ$. We will concentrate on lower order filters. Although higher order filters are commonly used in practice, many of the instrumentation components we use can be modeled as simple low order filters.

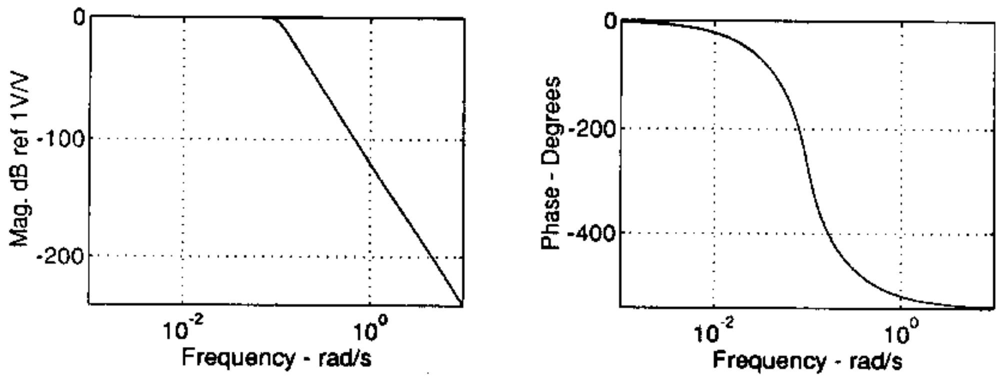

Figure 4: Magnitude and phase of the frequency response of a sixth order Butterworth filter.

High Pass Filters: filters that attenuate low frequencies

These filters are often used to remove low frequency noise from signals. Some instrumentation noise, known as flicker noise (see Chapter 8), has very high amplitudes at low frequencies. DC amplifiers (amplifiers that amplify both AC and DC components) tend to drift introducing low frequency components into the signals being measured. Some transducers naturally behave as high pass filters, e.g. piezoelectric accelerometers. The simplest high pass filter has a frequency response function of the form:

$$G(j\omega) = \frac{K\tau j\omega}{1+\tau j\omega} \tag{6}$$

Again K is the sensitivity, or *gain* and τ is still the time constant. The magnitude and phase are, assuming that K is positive:

$$|G(j\omega)| = \frac{|K\tau\omega|}{(1+\tau^2\omega^2)^{1/2}} \quad \text{and} \quad \arg(G(j\omega)) = \frac{\pi}{2} - \tan^{-1}(\tau\omega) \tag{7}$$

In general, we will assume that K is positive. If it is not, the negative sign just affects the phase, adding an additional π or $-\pi$ radians. The magnitude and phase are plotted in Figure 5.

The cut-off frequency is still defined as the point at which $|G(j\omega)|$ is $K/\sqrt{2}$. Note that here K is the gain of the frequency response at very high frequencies; when ω is large the "1+" part of the denominator becomes negligible and the transfer function tends to K. This is in contrast to low pass filters where we find K by examining what happens to the frequency response function as ω tends to zero. The cut-off frequency for this first order high pass filter is again $\omega_c = \tau^{-1}$. This was found by solving:

$$\frac{K^2\omega^2\tau^2}{1+\omega^2\tau^2} = \frac{K^2}{2} \tag{8}$$

The roll-off at very low frequencies, $\omega \ll \omega_c$, is 20 dB/decade because the frequency response function is approximately proportional to $K\tau\omega$ at very low frequencies. So for very low values of frequency, at one tenth of ω the magnitude has dropped by 0.1 from its value at ω, since $20\log_{10}(0.1) = -20\,\text{dB}$.

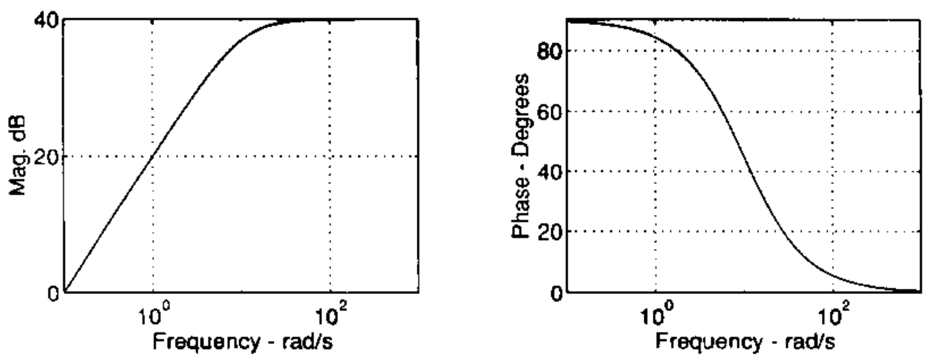

Figure 5: Magnitude and phase of a first order high pass filter

Shown below in Figure 6 is a simple RC circuit that behaves as a high pass filter. To derive the frequency response function, assume that no current flows to anything that the filter is connected to. The current flowing through the resistor and the capacitor is given by:

$$I = \frac{V_{out}}{R} = \frac{V_{in}}{R + \frac{1}{Cj\omega}} \tag{9}$$

By rearranging this expression the frequency response function is derived:

$$G(j\omega) = \frac{V_{out}}{V_{in}} = \frac{RCj\omega}{1+RCj\omega} \tag{10}$$

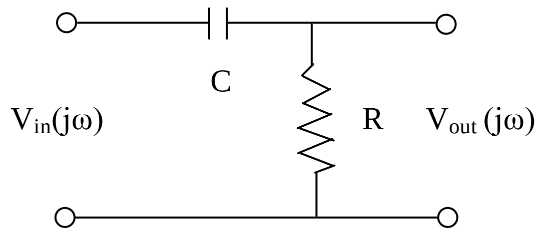

Figure 6: Simple high pass filter

The cut-off frequency, or 3 dB down point is $(RC)^{-1}$, and the sensitivity or gain of the filter is 1 (let $\omega \to \infty$ in the frequency response function to obtain this result).

Band Pass Filters: filters that attenuate both low and high frequency components in a signal.

An AC amplifier only amplifies components that lie in a certain frequency range, and attenuates all other frequencies; it therefore behaves as a band pass filter. A band pass filter can be created by putting a high pass filter in series with a low pass filter, although one must be careful in the design to avoid significant loading effects. A block diagram is shown in Figure 1. The simplest band pass filter one can create has a frequency response function:

$$G(j\omega) = K \frac{1}{1+\tau_1 j\omega} \frac{\tau_2 j\omega}{1+\tau_2 j\omega} \tag{11}$$

This has two cut-off frequencies that can be found by determining where the magnitude squared drops by a factor of a $\frac{1}{2}$. That is, from solving:

$$\frac{K^2}{2} = \frac{K^2 \tau_2^2 \omega^2}{\left(1+\tau_1^2\omega^2\right)\left(1+\tau_2^2\omega^2\right)} \tag{12}$$

If τ_2^{-1} and τ_1^{-1} are well separated then the solutions to equation (12) will be close to τ_1^{-1} and τ_2^{-1}, i.e., the cut-off frequencies of the low pass and the high pass filters, respectively. τ_2^{-1} and τ_1^{-1} is referred to as the **pass band** of the filter. The roll-offs at high and low frequencies, i.e., well above the higher cut-off frequency and well below the lower cut-off frequency, respectively, are both 20 dB/decade. In Figure 7 is shown the frequency response of the band pass filter. The mathematical expressions for the magnitude and phase of the frequency response function are:

$$\left|G(j\omega)\right| = \frac{\left|K\tau_2\omega\right|}{\left(1+\tau_1^2\omega^2\right)^{1/2}\left(1+\tau_2^2\omega^2\right)^{1/2}} \tag{13}$$

$$\text{Phase } G(j\omega) = \text{Phase}(j\tau_2\omega) - \text{Phase}(1+j\tau_2\omega) - \text{Phase}(1+j\tau_1\omega) \tag{14}$$

$$= \frac{\pi}{2} - \tan^{-1}\tau_2\omega - \tan^{-1}\tau_1\omega$$

Note that we have assumed that the sensitivity (gain in the flat part of the frequency response magnitude) is positive. If it is negative, then the magnitude is unchanged and the phase has an additional $+\pi$ or $-\pi$ term.

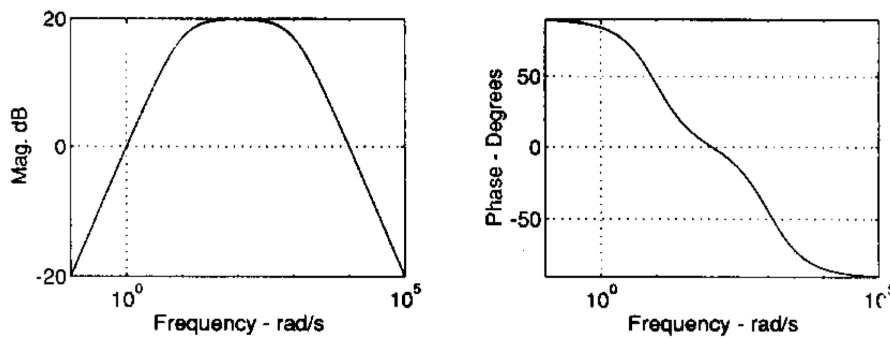

Figure 7: Magnitude and phase of a band pass filter,
cut-off frequencies 10 rad/s and 1000 rad/s, sensitivity or amplification, K = 10

If we combined the simple RC circuits shown in Figures 2 and 6, the frequency response function will not be the product of equations (5) and (10). With reference to Figure 8 one can show that the frequency response function of this circuit is:

$$G(j\omega) = \frac{V_{out}}{V_{in}} = \frac{R_2C_2j\omega}{(1+R_2C_2j\omega)(1+R_1C_1j\omega) + \frac{R_2}{R_1}R_1C_1j\omega} \tag{15}$$

The filter will be designed with certain cut-off frequencies in mind and so (R_1C_1) and (R_2C_2) will be set to be the inverse of the cut-off frequencies. You still have some flexibility in your choice of R_1 and R_2. By choosing $R_1 \gg R_2$, the last term in the denominator becomes small, and the frequency response function becomes close to the product of equations (5) and (10).

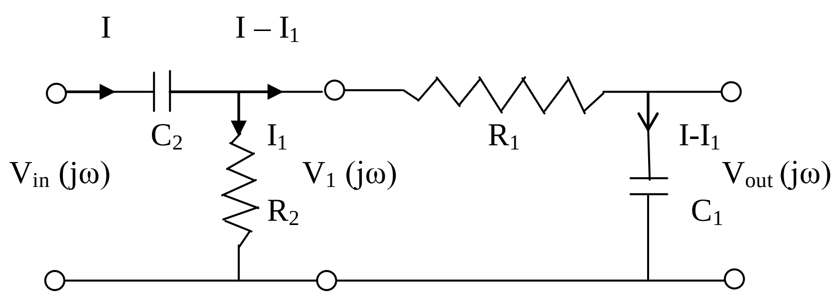

Figure 8: A high pass and a low pass filter in series.

Notch or Band Stop Filters: filters that attenuate only a range** of **frequencies

A notch filter can be created by passing the signal through a low pass filter and through a high pass filter simultaneously and summing the output of the two filters. In Figure 9 is shown a schematic of this parallel arrangement of filters. These filters are useful for removing line noise from signals (60Hz).

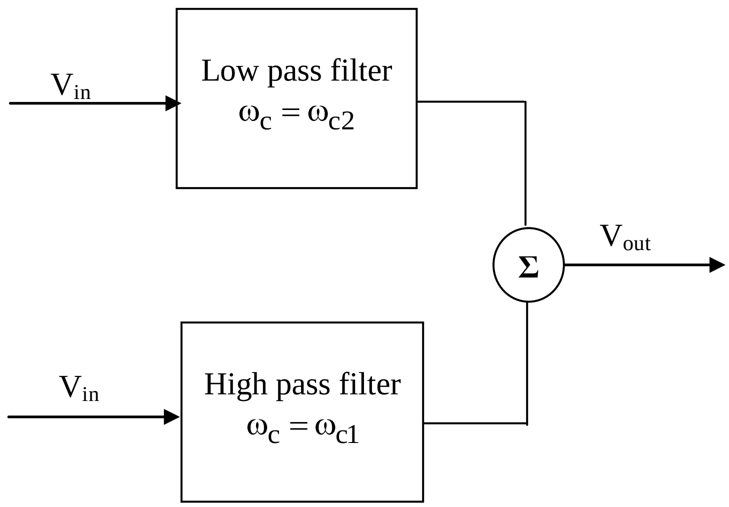

Figure 9: A notch or band stop filter

If the range of frequencies that you wish to remove from the signal is very small then you will, in general, need to use high order filters (polynomial in ω in the denominator will contain higher order terms) so that the cut-off rate is high. If you use low order filters, when you sum the signals from the low and high pass filters there may be little difference between the frequency content of the notch filter's output and the input signals. This is because the roll-off region of the low pass filter overlaps significantly with the roll-off region of the high pass filter, when the filter orders are low and the range of frequencies that we wish to remove is small. In Figure 10(a) is shown the magnitude of the frequency response of a low order notch filter, and in Figure 10(b) is shown the magnitude of the frequency response of a high order notch filter.

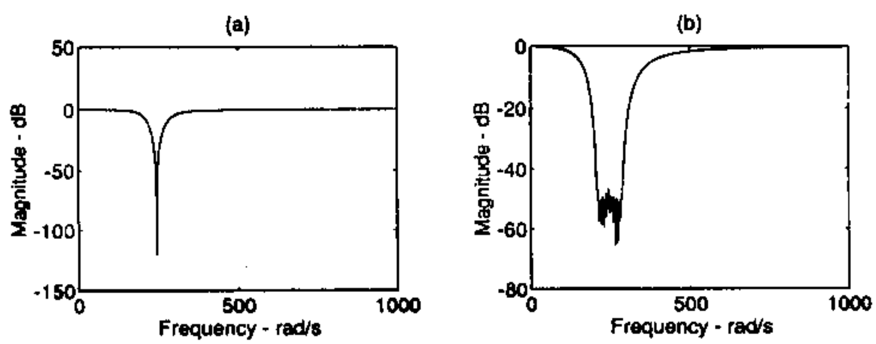

Figure 10: The magnitude of the frequency response of two band stop (notch) filters. (a) Low order filter, (b) high order filter

In this section we have looked at four types of simple filters. In a sense any system is a filter and many transducers behave like low, high, or band pass filters. We often construct filters in measurement systems to control the effects of noise. Noise in control systems can cause interesting problems when trying to implement a controller that has been designed without consideration of noise effects. The point at which the noise enters the controlled system is also important. Judicious use of filters in these systems is essential for robust control.

LOADING

When considering the interaction effects of systems it is useful to model a voltage source as an ideal source in series with a single resistor R_s. Unfortunately, very few dynamic systems come to us in precisely such a form. Thevenin's Theorem prescribes a method for obtaining the value of E_{th} (source voltage) and R_{th} (source impedance), which then permits us to replace the rather complicated system with the above described *equivalent* circuit. A paraphrase of Thevenin's Theorem is as follows:

> *In so far as a load is concerned, any one-port network of linear elements and energy sources can be replaced by a series combination of an ideal voltage source* E_{th} *and a linear impedance* Z_{th} *where* E_{th} *is the open-circuit voltage of the one-port network and* Z_{th} *is the ratio of the open-circuit voltage to the short-circuit current.*

The corresponding one-port circuit can be reduced to an equivalent representation of the form:

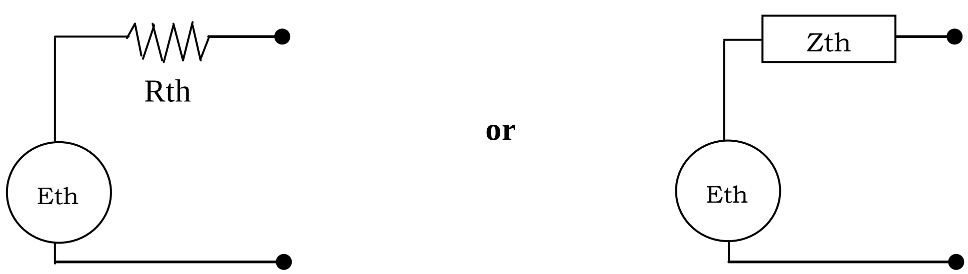

Figure 11: Equivalent one port representations

The term "one-port" simply means a single pair of terminals from which energy flows. Thus the *Thevenin* resistance R_{th} can be thought of as the output impedance of the source and is a measure of how much voltage drop the source will experience as measured from the open circuit condition as more and more current flows from the source. If the original circuit being investigated contains capacitors and/or inductors, we have to employ complex impedance notation to write the expression for Z_{th}. Finally, if the load is represented by a single equivalent impedance, R_L or Z_L, we have the source-load equivalent representation shown below:

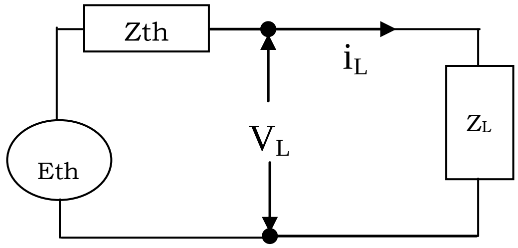

Figure 12

By employing a voltage divider concept we can quickly determine the frequency response function between E_{th} and V_L:

$$\frac{V_L}{E_{th}}(j\omega) = \frac{Z_L}{Z_{th} + Z_L} = \frac{1}{1 + Z_{th}/Z_L} \tag{16}$$

Note that $|V_L|$ will always be equal to or less than $|E_{th}|$.

If all the voltage sources are independent (batteries for example) as opposed to dependent (amplifiers for example) the task of determining Z_{th} and E_{th} is relatively simple.

Example

Suppose we have:

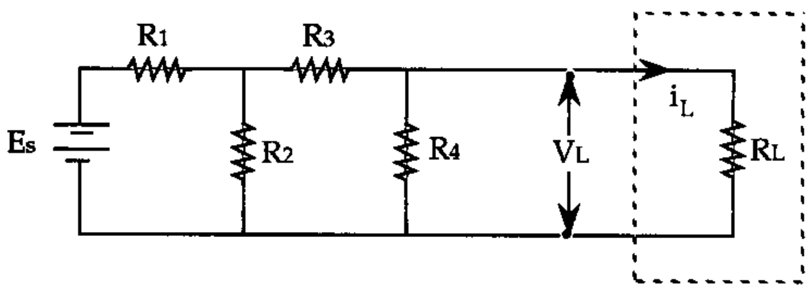

Figure 13:

The Thevenin resistance is found by short circuiting the voltage source and finding the resistance looking in at the output port. If the battery E_S is known to have internal resistance, another resistor must be place in series with the battery.

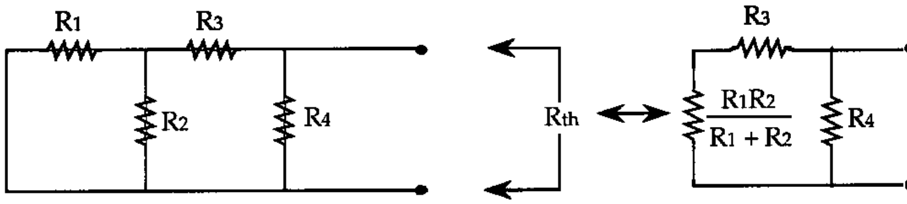

Figure 14:

$$R_{th} = \frac{\left(R_3 + \frac{R_1 R_2}{R_1 + R_2}\right) R_4}{R_3 + \frac{R_1 R_2}{R_1 + R_2} + R_4} \tag{17}$$

The Thevenin source voltage is found by disconnecting the load and measuring, or calculating, the open circuit voltage across the two terminals where the load will be connected.

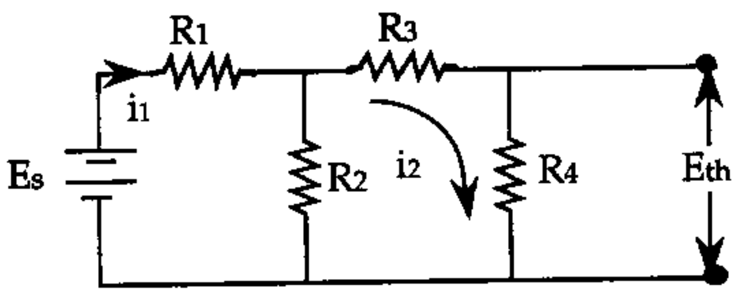

Figure 15:

Writing simultaneous equations for the two loops:

$$R_1 i_1 + R_2 (i_1 - i_2) = E_s \tag{18a}$$

$$R_2 (i_2 - i_1) + R_3 i_2 + R_4 i_2 = 0 \tag{18b}$$

Solving for i_1 from (18b):

$$i_1 = \frac{R_2 + R_3 + R_4}{R_2} \tag{19a}$$

Substituting into (18a) gives:

$$\frac{(R_2 + R_3 + R_4)(R_1 + R_2)}{R_2} i_2 - R_2 i_2 = E_s$$

$$i_2 = \frac{R_2}{R_1 (R_2 + R_3 + R_4) + R_2 (R_3 + R_4)} E_s \tag{19b}$$

and

$$E_{th} = R_4 i_2 \tag{20}$$

Finally, the actual load voltage can be quickly determined by using the equivalent circuit

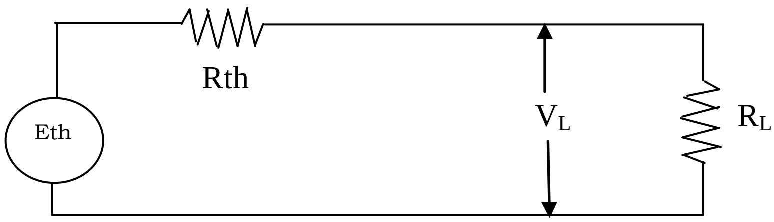

Figure 16:

$$V_L = \frac{R_L}{R_L + R_T} E_{th} = \frac{1}{1 + R_T/R_L} E_{th} \tag{21}$$

Example

Consider the case of a first order low pass filter being driven by a source with nonzero output impedance and being loaded by a readout instrument with non-infinite input impedance:

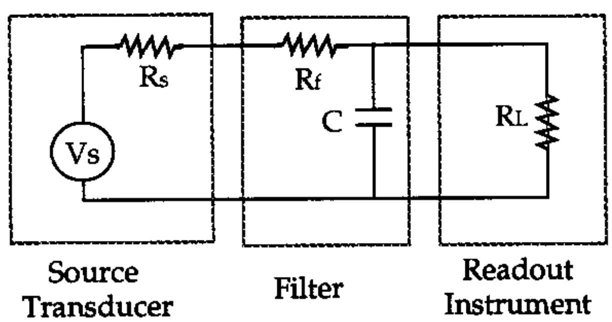

Figure 17:

A very simple approach to the design of the filter stage would be to assume $R_L = \infty, R_s = 0$ and choose R_f and C so that the filtering requirements are met. If this simple approach is adopted, it will be noted that only the product $R_f C$ will be specified, with no rationale for choosing R_f and C individually. Let us now assume nonideal values for R_s and R_L: that is, $R_s > 0$, R_L is large but finite. Applying Thevenin's Theorem:

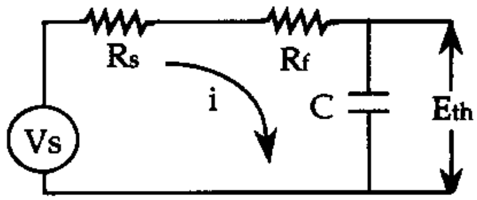

Figure 18

$$E_{th} = I \cdot \frac{1}{Cj\omega} = \frac{V_s}{R_s + R_f + \frac{1}{Cj\omega}} \cdot \frac{1}{Cj\omega} = \frac{V_s}{(R_s + R_f)Cj\omega + 1} \tag{22}$$

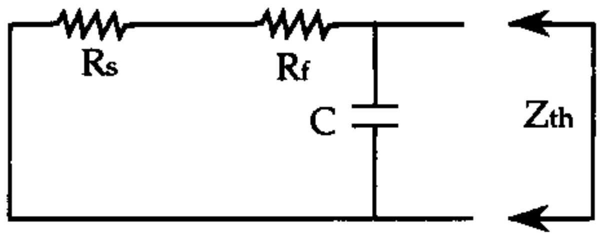

Figure 19:

$$Z_{th} = \frac{R_s + R_f \cdot \frac{1}{Cj\omega}}{R_s + R_f + \frac{1}{Cj\omega}} = \frac{R_s + R_f}{(R_s + R_f)Cj\omega + 1} \tag{23}$$

The equivalent circuit becomes:

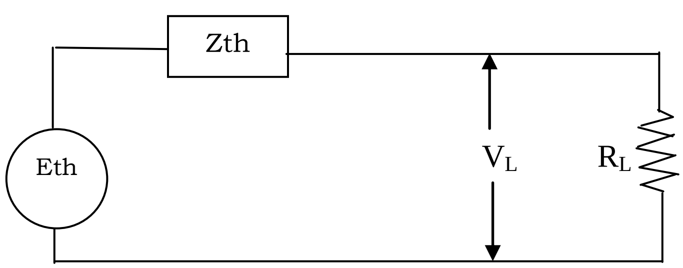

Figure 20:

$$V_L = \frac{R_L}{Z_{th} + R_L} E_{th} = \frac{R_L}{\frac{R_s + R_f}{(R_s + R_f)Cj\omega + 1} + R_L} \frac{V_s}{(R_s + R_f)\, Cj\omega + 1} \tag{24}$$

$$\frac{V_L}{V_s}(j\omega) = \frac{\dfrac{1}{1+(R_s+R_f)/R_L}}{\dfrac{(R_s+R_f)}{1+(R_s+R_f)/R_L}Cj\omega+1} = \frac{K}{\tau j\omega+1} \tag{25}$$

where

$$K = \frac{1}{1+\dfrac{R_s+R_f}{R_L}} \qquad \tau = \frac{(R_s+R_f)C}{1+\dfrac{R_s+R_f}{R_L}}$$

If we consider the ideal case where $R_S = 0$ and $R_L = \infty$, we see that $K_{ideal} = 1$ and $\tau_{ideal} = R_f C$. As the value of R_L decreases, the static gain also decreases but the equivalent cut-off frequency, $\omega_c = 1/\tau$, increases. If R_S increases from zero, the static sensitivity decreases and the cut-off frequency also decreases. Thus the conclusion is that when inserting an intermediate stage of a signal processing circuit, one must be cognizant of both the preceding stage output impedance as well as the succeeding stage input impedance.

In general, when one is connecting several subsystems, whose individual frequency response functions are $G_i(j\omega)$, the frequency response function of the connected system is:

$$G(j\omega) = \left[\prod_{i=1}^{N-1} G_i(j\omega)L_{i,i+1}\right]G_N(j\omega) \tag{26a}$$

where

$$L_{i,i+1} = \frac{Z_{in_{i+1}}}{Z_{in_{i+1}} + Z_{out_i}} \tag{26b}$$

$Z_{in_{i+1}}$ is the input impedance of the (i + l)th subsystem, Z_{out_i} is the output impedance of the (i)th subsystem and N is the total number of subsystems connected together.

Example

A flow meter is connected to an amplifier which is connected to a tape recorder. The output impedance of the flow meter is 200Ω. The input and output impedances of the amplifier are 100,000Ω and 150Ω, respectively. The input impedance of the tape recorder is 2000Ω.

a. Calculate the loading terms: L_{12} and L_{23}.

b. The flow meter is outputting 0.02 Volts before connecting it to the amplifier. The amplifier has a constant frequency response function gain of ×300 with zero phase. What is the input voltage to the tape recorder when everything is connected?

Solution

a.

$$L_{12} = \frac{Z_{in_{AMPLIFIER}}}{Z_{in_{AMPLIFIER}} + Z_{out_{FLOWMETER}}} = \frac{100,000}{100,000 + 200} = 0.998$$

$$L_{23} = \frac{Z_{in_{TAPE}}}{Z_{in_{TAPE}} + Z_{out_{AMPLIFIER}}} = \frac{2,000}{2,000 + 150} = 0.952$$

b. If there were no loading effects the voltage into the tape recorder would be:

$$V_{in_{TAPE-IDEAL}} = 0.02 \times 300 = 6 \text{ Volts}$$

With the loading the input voltage will be:

$$V_{in_{TAPE-IDEAL}} \times L_{12} \times L_{23} = 6 \times 0.998 \times 0.952 = 5.703 \text{ Volts}$$

Example

A thermocouple has a time constant of 0.025 seconds and a sensitivity of 0.022 Volts/°C. It has an output impedance of 500 Ω. It is connected to a high-pass filter with a frequency response function:

$$G_{FILTER}(j\omega) = \frac{j2\omega}{1 + j0.02\omega}$$

and an input impedance of 5000Ω.

What is the frequency response function relating the output of the filter to the input of the thermocouple?

Solution

The frequency response function of the entire system is:

$$G_{TOTAL} = G_{THERMO}\, G_{FILTER}\, L_{12}$$

$$= \frac{0.022}{1 + j\omega 0.025} \frac{j2\omega}{1 + j0.02\omega} \frac{5000}{5000 + 500} = \frac{j0.04\omega}{(1 + j0.025\omega)(1 + j0.02\omega)}$$

OPERATIONAL AMPLIFIERS

Operational amplifiers have found their way into a large number of applications in the fields of instrumentation, data acquisition, communications, consumer electronics and control systems, to name but a few. An operational amplifier, generally referred to as an *op amp,* is essentially a high gain, zero to high frequency, differential input amplifier. It derives its name from the fact that it can be made to perform a variety of mathematical operations, such as integration, differentiation, addition, subtraction, etc.

Once the basic equations are understood, op amps are relatively easy to use and help make many circuit design problems straightforward. Probably, an op amp's most fundamental properties are its very large input impedance and its low output impedance. Its very high input impedance capability allows the designer to cascade stages of passive circuitry with virtually no current flowing between stages. In good subsystem design the $L_{i,i+1}$ terms in equation (26) are approximately equal to 1. Therefore each stage can be isolated and analyzed separately. Furthermore, the low output impedance of the op amp makes it possible to drive a subsystem having a relatively low input impedance.

Operating Characteristics

An op amp has two input signal terminals and one output signal terminal as well as positive and negative voltage supply terminals and one or more external trim terminals to eliminate unwanted DC offset.

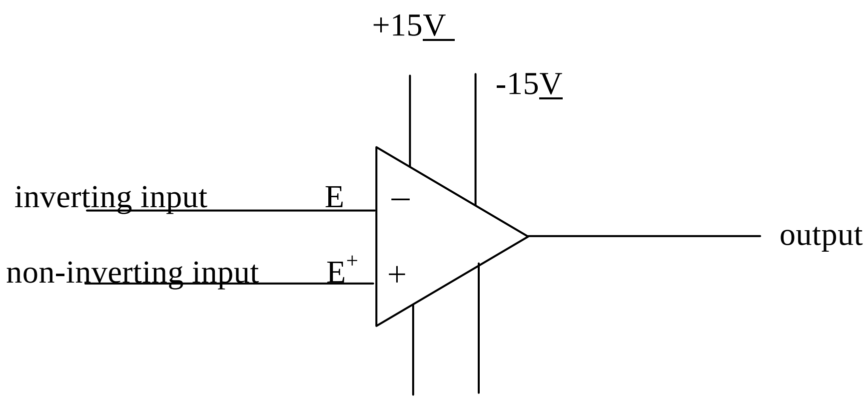

Figure 21: Trim Ports

The two signal input ports, called the inverting (-) and the non-inverting (+) inputs, can have input impedances as high as thousands of megohms. The output impedance is typically 100 ohms or less. Its typical input-output static characteristics are as shown in Figure 22.

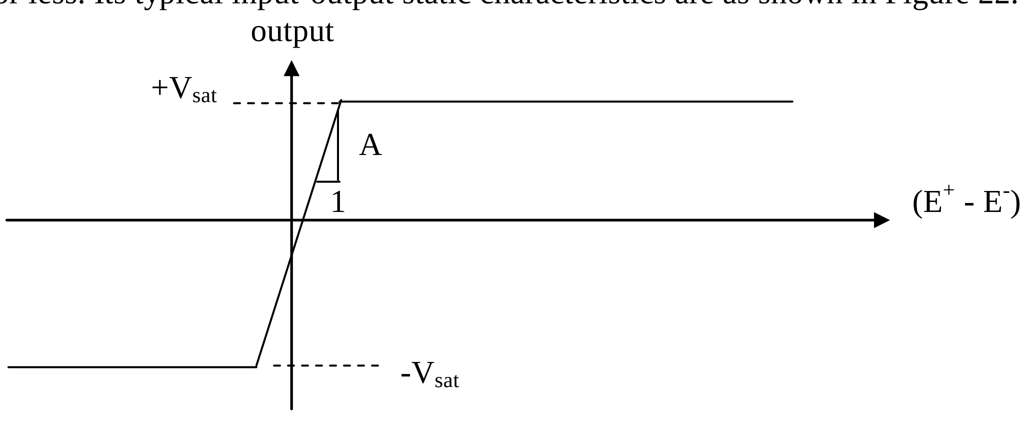

Figure 22: Calibration curve for an op amp

V_{sat} is the saturation voltage. For a +15 volt supply, V_{sat} is approximately 13 volts. The value of the gain A is approximately 100,000 Volts/Volt. Because the <u>open loop</u> gain is so large, an op amp is rarely used without external components such as resistors and capacitors. The gain A is very frequency dependent, as shown below in Figure 23.

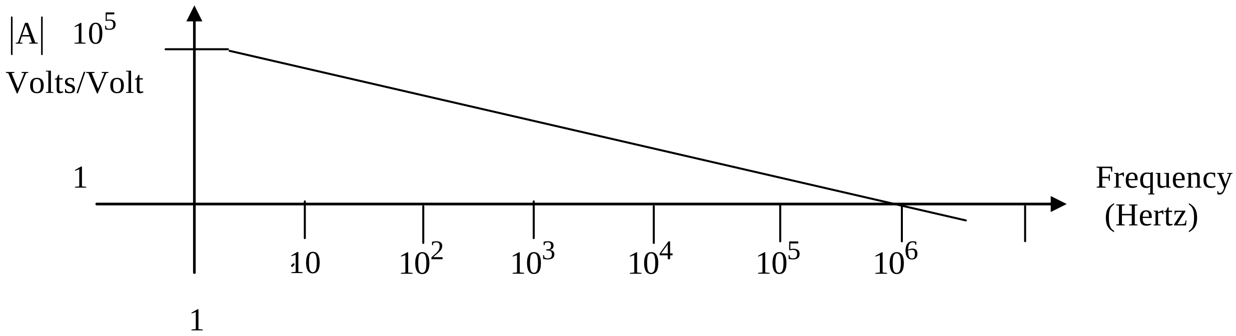

Figure 23: Gain versus frequency of a typical op amp

Mathematical Transfer Function Derivation

The most commonly encountered op amp circuit can be represented as follows:

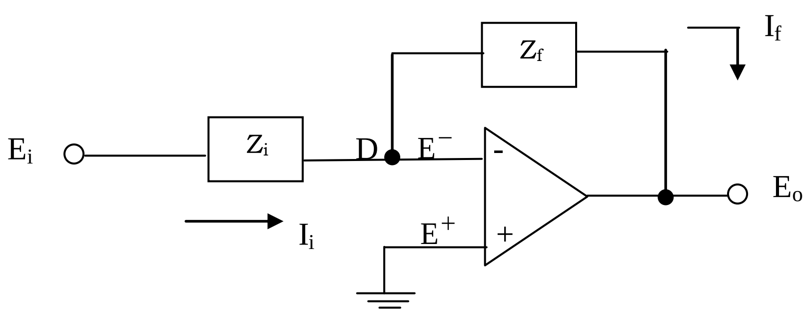

Figure 24: Basic op amp configuration with feedback and input impedances.

Using complex amplitude notation, i.e., relating the frequency domain or s-plane characteristics of the voltages, impedances and currents, we have:

$$\frac{E_i - E^-}{Z_i} = I_i \qquad (27)$$

and

$$\frac{E^- - E_o}{Z_f} = I_f \qquad (28)$$

Because the input impedance of an op amp is very large, almost zero current flows through the op amp. Therefore, it can be assumed that the input current equals the feedback current, $i_i = i_f$, giving:

$$\frac{E_i - E^-}{Z_i} = \frac{E^- - E_o}{Z_f} \qquad (29)$$

Up to this point we have not made use of the high gain relationship of the op amp, namely: $E_o = A(E^+ - E^-) = -AE^-$ because E^+ is at ground. Substituting for E^- in the external circuit equation:

$$\frac{E_i - \left(-\dfrac{E_o}{A}\right)}{Z_i} = \frac{\left(-\dfrac{E_o}{A}\right) - E_o}{Z_f} \tag{30}$$

Solving for the input-output relationship we obtain:

$$\frac{E_o}{E_i} = \frac{-1}{\dfrac{Z_i}{Z_f}\left(\dfrac{1}{A}+1\right)+\dfrac{1}{A}} \tag{31}$$

As was illustrated in Figure 23, the magnitude of the open loop frequency response function A is very much greater than unity over a wide range of frequencies. Therefore the input-output relationship becomes:

$$\frac{E_o}{E_i} = -\frac{Z_f}{Z_i} \tag{32}$$

If we wished to amplify a signal (and unavoidably invert the sign), we replace Z_f and Z_i by resistors and we have:

$$\frac{e_o(t)}{e_i(t)} = -\frac{R_f}{R_i} \tag{33}$$

(Note: we use upper-case letters to denote voltages in the frequency domain, as functions of ω, and lower-case letters in the time domain.)

You may notice that the result derived by letting (1/A) tend to zero because A is large, could have been derived by assuming that the voltage at both the inverting and non-inverting terminals of the op-amp are equal. Since the non-inverting terminal is connected to ground here, the voltage at both terminals would be 0 Volts. Again assuming that virtually no current flows through the op amp, summing currents flowing into junction D, and setting the result to 0 (Kirchoff's Law) will also give equation (32).

General Rules for Obtaining Transfer Functions of Circuits Containing Op Amps

At this point we can give two simple rules that allow us to incorporate the effect of op amps in a circuit.

Rule 1 The voltages at the inverting and non-inverting input terminals are equal.
Rule 2 No current flows through the op amp.

A Summing and an Integrating Circuit

The circuit schematic for summing a number of signals is shown in Figure 25.

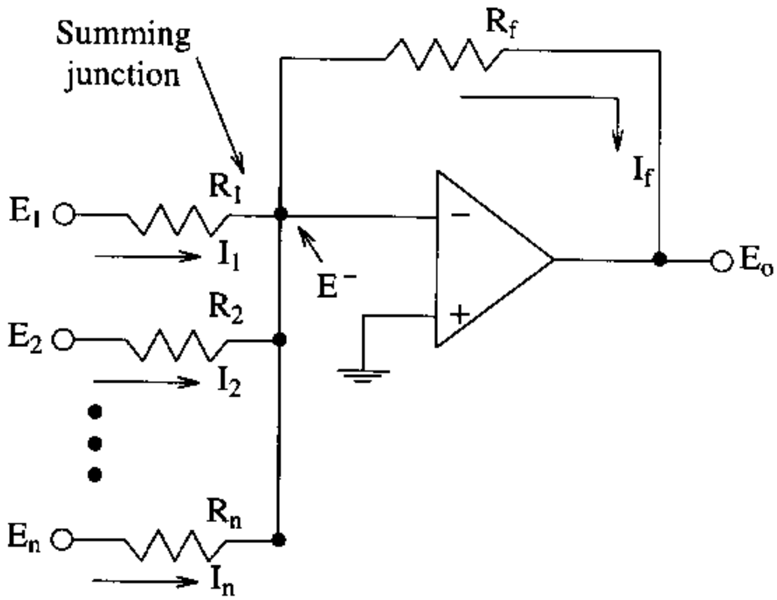

Figure 25: Summing Circuit

Assume $e^- = 0$, because the non-inverting input terminal is connected to ground, and that no current flows through the op amp. Summing the currents at the inverting terminal node, called the summing junction yields:

$$\frac{e_1}{R_1} + \frac{e_2}{R_2} + \cdots + \frac{e_n}{R_n} = -\frac{e_o}{R_f} \tag{34}$$

Hence,

$$e_o = -\sum_{i=1}^{n} \frac{R_f}{R_i} e_i$$

Note that each input voltage is amplified by the ratio of the common feedback resistor and respective input resistor. It is worth repeating that the summing junction sums currents, not voltages. The voltage at the summing junction is very nearly zero.

The operation of integration is easily performed by inserting a capacitor in place of Z_f. We have:

$$E_o = -\frac{1}{Cj\omega} \sum_{i=1}^{n} \frac{E_i}{R_i} \tag{35}$$

In the time domain:

$$e_o(t) = -\frac{1}{C}\int_o^t\left(\sum_{i=1}^{n}\frac{e_i}{R_i}\,dt\right) + e_o(0) = -\sum_{i=1}^{n}\frac{1}{R_iC}\int_o^t e_i(t)dt + e_o(0) \qquad (36)$$

Each incoming voltage is amplified by the factor $\frac{1}{R_iC}$ as well as integrated. Again the output signal is inverted.

Impedance Buffering

Very often it is important to prevent the flow of current from a circuit, and hence the subsystem to which this circuit is connected must have a high input impedance. If the output impedance of a preceding stage is very large, a relatively small amount of current will cause a relatively large voltage attenuation due to loading. A follower circuit is an active circuit which prevents current from flowing but the output voltage of the follower circuit equals the voltage across the preceding stage output terminals when nothing is connected to it. This follower circuit is especially simple:

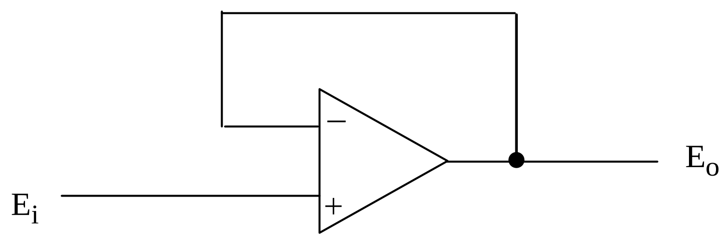

Figure 26: Follower circuit

Using the op amp open loop gain relations:

$$e_o = A(e^+ - e^-) = A(e_i - e_o)$$

$$\Rightarrow \quad e_o = \frac{A}{1+A}e_i \approx e_i \qquad (37)$$

Differential Amplifier

Many instrument transducers generate two voltages which may be combined linearly into a single voltage, by using the following circuit.

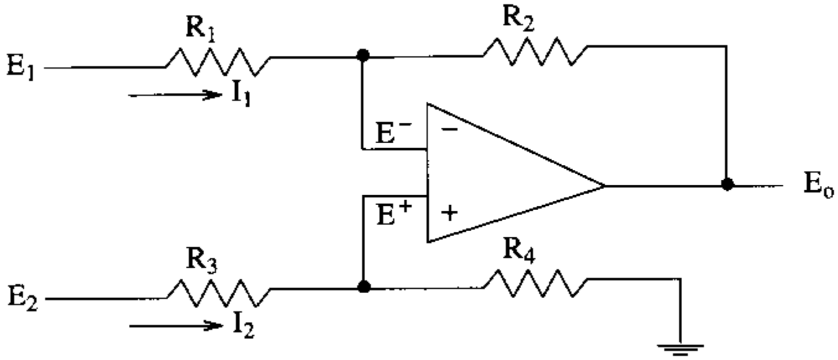

Figure 27: Differential amplifier

Because they are equal, lets denote $e^+$and e^- by e_A. Summing currents flowing into the junctions closest to the input terminals of the op amp yields:

$$\frac{e_1 - e_A}{R_1} + \frac{e_o - e_A}{R_2} = 0$$

and

$$\frac{e_2 - e_A}{R_3} + \frac{0 - e_A}{R_4} = 0$$

Rearranging one can express e_A as a function of e_1 and e_o or as a function of e_2.

$$e_A = \frac{R_2}{R_1 + R_2}\, e_1 + \frac{R_1}{R_1 + R_2}\, e_o = \frac{R_4}{R_3 + R_4}\, e_2$$

and thus:

$$e_o = \frac{R_1 + R_2}{R_1}\left[\frac{R_4}{R_3 + R_4}\, e_2 - \frac{R_2}{R_1 + R_2}\, e_1\right]$$

Furthermore, if $R_1 = R_3$ and $R_2 = R_4$:

$$e_o = \frac{R_2}{R_1}\;\; e_2 - e_1 \tag{38}$$

That is, the circuit amplifies the difference between e_2 and e_1, hence it is called a differential amplifier.

ACTIVE FILTER DESIGN

As we discussed in the loading section joining systems together can cause problems because of interaction effects. This can be solved by designing systems with low output impedances and high input impedances. It is almost impossible to achieve this consistently with passive filters. We therefore use operational amplifiers, either as impedance buffers or as an integral part of the filter circuit. Both approaches will be demonstrated in this section. An added advantage of using active devices is the capability of increasing the static gain of the filter. The term active is used because the op amps themselves require their own power supply, whereas the filters described earlier in the chapter are passive containing only impedances.

An Active Low Pass Filter

Consider the following circuit:

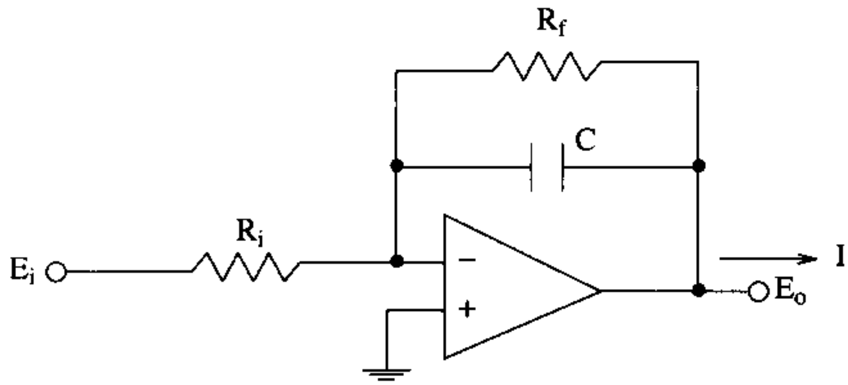

Figure 28: Low pass filter

By using the two rules governing op amp operation (see page 7-18) we can show that:

$$\frac{E_o}{E_i} = -\frac{Z_f}{Z_i} = -\frac{\frac{R_f}{R_f C j\omega + 1}}{R_i} = -\frac{R_f}{R_i}\frac{1}{\tau j\omega + 1} \tag{39}$$

where $\tau = R_f C$. Notice that the gain can be made larger than unity. The input impedance of this circuit is effectively R_i, which may or may not be adequately large. If R_i is too large, either R_f must be exorbitantly large or the static gain must be restricted. Op amps have an upper limit on resistive values used in their input and feedback networks. If these resistive values are too large, the offset voltages, due to imperfections in the op amp operation, become large also.

A second circuit which has a much higher input impedance is that of a low pass filter connected to the non-inverting terminal of an op amp as shown in Figure 29.

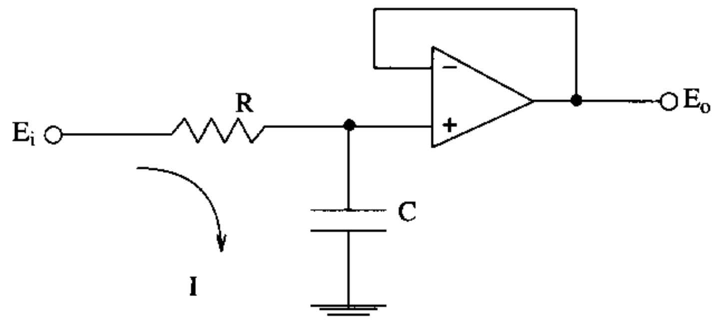

Figure 29: Passive low pass filter plus follower

The frequency response function of this circuit is:

$$\frac{E_o}{E_i}(j\omega) = +\frac{1}{\tau j\omega + 1} \quad , \quad \tau = RC \tag{40}$$

The output of the filter terminates in the high impedance of the non-inverting input of the op amp, and so for frequencies well below the cut-off frequency, $\omega \ll (RC)^{-1}$, the input impedance as seen by e_i is very large. It is, however, limited by the series impedance $R + \frac{1}{Cj\omega}$. The output impedance of the follower is essentially that of the op amp, i.e., low, and hence the input impedance of the next stage is much higher than it is.

A perfectly buffered low pass filter circuit is shown in Figure 30. This has unity gain in the pass band of the filter. The gain of the circuit can be increased by employing a follower with a circuit of the form shown in Figure 31.

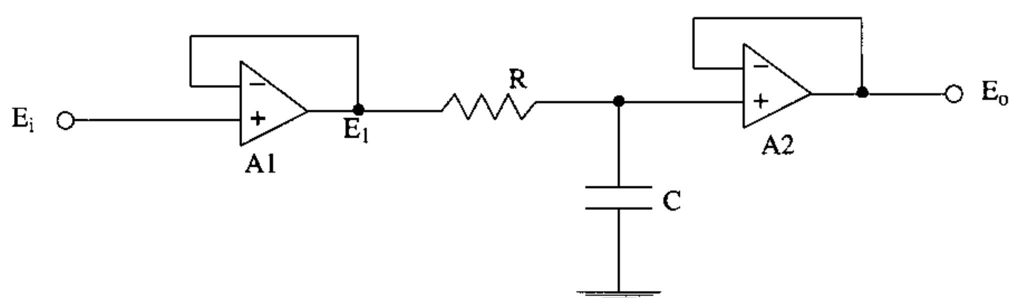

Figure 30: Perfectly buffered low pass filter

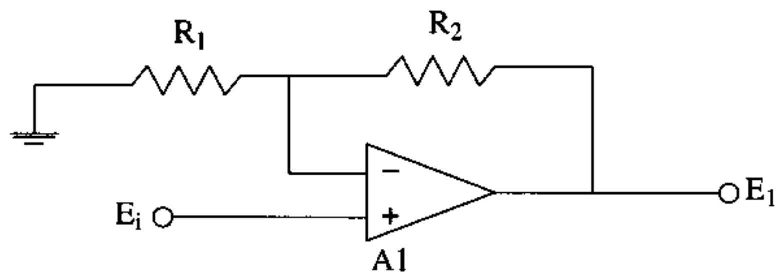

Figure 31: Combined amplifier and buffer

$$\frac{e_1}{e_i} = 1 + \frac{R_2}{R_1}$$

hence

$$\frac{e_o}{e_i} = \frac{1 + R_2 / R_1}{\tau j\omega + 1} \tag{41}$$

An Active Band Pass Filter

A bandpass network can be synthesized as follows:

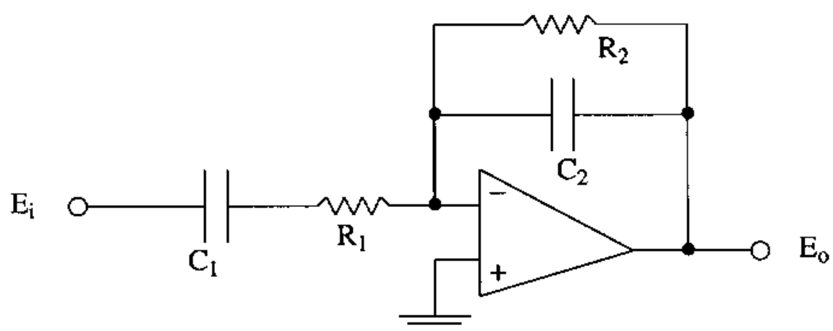

Figure 32: Bandpass Filter

$$\frac{E_o}{E_i}(j\omega) = -\frac{Z_f}{Z_i} = -\frac{\dfrac{R_2}{R_2 C_2 j\omega + 1}}{R_1 + \dfrac{1}{C_1 j\omega}} = -\frac{R_2 C_1 j\omega}{(\tau_1 j\omega + 1)(\tau_2 j\omega + 1)}$$

$$= -\frac{R_2}{R_1} \frac{\tau_1 j\omega}{(\tau_1 j\omega + 1)(\tau_2 j\omega + 1)} \tag{42}$$

Again it can be seen that the amplitude ratio in the pass band can be made greater than unity by adjusting R_2 / R_1.

Another circuit which employs a follower amplifier with gain to prevent loading between stages and at the output is:

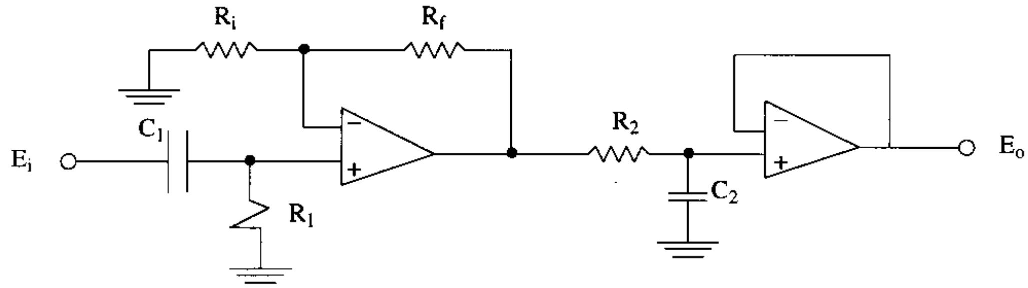

Figure 33: Bandpass filter

$$\frac{E_o}{E_i}(j\omega) = \left(1 + \frac{R_f}{R_i}\right)\frac{\tau_1 j\omega}{(\tau_1 j\omega + 1)(\tau_2 j\omega + 1)} \tag{43}$$

where $\tau_1 = R_1C_1, \tau_2 = R_2C_2$. Amplifier A1 serves as an impedance buffer between the high pass and low pass stages as well as increasing the gain in the pass band.

SUMMARY

In this chapter we have considered the design of filters and the problems that occur when connecting subsystems together. In particular, loading effects cause the frequency response function of the system to no longer be equal to the product of the frequency response functions of the subsystems; each connection introduces a loading term. In order to reduce loading effects we need to incorporate op amps into the design of the subsystems. With the use of op amps we can design subsystems with low output impedances and high input impedances, the loading terms tend to one and the frequency response function of the connected system is approximately the product of the frequency response functions of the individual subsystems.

Op amps have high input impedances and low output impedances, although one should be careful not to undermine their high input impedances by connecting low impedances to the input terminals. In the basic configuration (see Figure 24) the frequency response function is $-Z_f / Z_i$ and the input impedance is Z_i, because in this configuration the input inverting terminal is at 0 Volts, and is termed a virtual ground. The input voltage then sees the impedance Z_i when looking into the op amp circuit. There are two basic rules for op amps, which should be used when deriving the frequency response functions of circuits with op amps. First: no current flows through the op amp and secondly: the voltages at the inverting and non-inverting inputs terminals are equal.

The last part of the chapter was a brief introduction to the use of op amps in active filter design.