4.0 Two-Way ANOVA

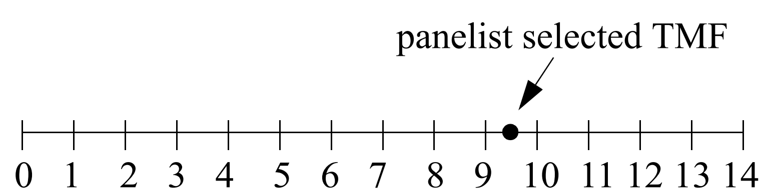
(Updated Spring 2005)

Design of Experiment techniques are widely applicable within engineering and the sciences. We will now consider an example that comes from the Food Science area.

Hot Dog Example

Recent Federal Regulations have relaxed requirements on salt, protein, fat, and filler contents in hot dogs (or frankfurters). There is a demand for healthier food products (less salt, less fat). With this in mind, manufacturers are interested in developing low fat/salt hot dogs. Reducing the fat and salt levels within a hot dog may create a problem since these substances tend to bind the water in the product. This may result in mushy hot dogs that consumers do not like.

Consumer acceptance of hot dogs with new formulations are perhaps best assessed with "Texture Mouth Feel." This measure of performance (response) is assessed by asking a panelist to judge the texture (TMF) on a 0 to 14 scale (0: soft/mushy, 14: hard/chewy). A typical scale is shown below -- the TMF is measured to the nearest tenth of a unit.



Scale for grading the TMF value of a hot dog

In preparation for the hot dog experiment, three panelists were trained to evaluate products consistently. Four hot dog formulations were created that is was of interest to evaluate. The hot dog experiment therefore considered the following variables (and is focused on answering the indicated questions):

- Hot Dog Formulation -- 4 levels (is there a texture difference between formulations?)
- Panelist -- 3 levels (are they consistent with one another?)

Several hot dogs of each formulation were given to the panelists in random order.

Hot Dog Formulation						
		A	B	C	D	
Panelists	1	7.6 6.5 7.2	11.4 7.6 9.5	6.3 7.9 6.8	2.7 3.1 1.7	$\bar{\bar{y}}_1 = 6.52$
	2	7.2 13.6 10.7	12.9 12.4 10.7	11.1 6.9 9.0	3.3 1.9 2.3	$\bar{\bar{y}}_2 = 8.50$
	3	7.0 10.2 8.3	10.2 8.1 8.7	6.8 9.2 11.0	3.7 2.2 3.1	$\bar{\bar{y}}_3 = 7.38$
		$\bar{\bar{y}}_A = 8.70$	$\bar{\bar{y}}_B = 10.16$	$\bar{\bar{y}}_C = 8.33$	$\bar{\bar{y}}_D = 2.67$	$\bar{\bar{\bar{y}}}$ **=7.47**

$i = 1, p$ (p is the number of panelists) $p = 3$
$j = 1,h$ (h is the number of hot dog forms) $h = 4$
k = $1,r$ (r is the number of replicates) $r = 3$
y_{ijk} = texture for ith panelist, jth hot dog, kth replication.

We can calculate a mean for each cell (hot dog - panelist combination)

$$\bar{y}_{ij} = \sum_{k=1}^{r} y_{ijk}/r$$

We can calculate a mean for each panelist and hot dog.

$$\bar{\bar{y}}_i = \sum_{j=1}^{h} \sum_{k=1}^{r} y_{ijk}/(rh)$$

$$\bar{\bar{y}}_j = \sum_{j=1}^{p} \sum_{k=1}^{r} y_{ijk}/(rp)$$

$$\bar{\bar{\bar{y}}} = \frac{\sum_{i=1}^{p} \sum_{j=1}^{h} \sum_{k=1}^{r} y_{ijk}}{phr}$$

The Cell Averages, $\bar{y}_{ij}$, are listed in the following table,

Cell Average

	A	B	C	D
1	7.1	9.5	7.0	2.5
2	10.5	12.0	9.0	2.5
3	8.5	9.0	9.0	3.0

We can calculate an estimate of the variance, σ_y^2, for each panelist - hot dog combination. $s_{1A}^2, s_{1B}^2, s_{1C}^2, \ldots, s_{3D}^2$ all with *r - 1* degrees of freedom (DOF).

$$s_{ij}^2 = \sum_{k=1}^{r} (y_{ijk} - \bar{y}_{ij})^2/(r-1)$$

The pooled variance is then:

$$s_{pe}^2 = \frac{S_{PE}}{\nu_e} = \frac{\sum_{i=1}^{p}\sum_{j=1}^{h}(r-1)s_{ij}^2}{hp(r-1)} = \frac{\sum_{i=1}^{p}\sum_{j=1}^{h}\sum_{k=1}^{r}(y_{ijk}-\bar{y}_{ij})^2}{hp(r-1)}$$

Use the variation of the hot dog means about $\overline{\overline{\overline{y}}}$ to estimate σ_y^2 by S_H.

Use the variation of the panelist means about $\overline{\overline{\overline{y}}}$ to estimate σ_y^2 by S_P

$$S_{TOT} = \sum_{i=1}^{p}\sum_{j=1}^{h}\sum_{k=1}^{r} y_{ijk}^2,\ S_{AVG} = phr(\overline{\overline{\overline{y}}})^2$$

Decompose the variation in the data:

$$y_{ijk} = \overline{\overline{\overline{y}}} + (\overline{\overline{y}}_i - \overline{\overline{\overline{y}}}) + (\overline{\overline{y}}_j - \overline{\overline{\overline{y}}}) + (y_{ijk} - \bar{y}_{ij}) + (\bar{y}_{ij} + \overline{\overline{\overline{y}}} - \overline{\overline{y}}_i - \overline{\overline{y}}_j)$$

Squaring both sides and summing over reps, hot dogs, and panelists.

$$\sum_{i=1}^{p}\sum_{j=1}^{h}\sum_{k=1}^{r} y_{ijk}^2 = rph(\overline{\overline{\overline{y}}})^2 + hr\sum_{i=1}^{p}\left(\overline{\overline{y}}_i - \overline{\overline{\overline{y}}}\right)^2 + pr\sum_{j=1}^{h}\left(\overline{\overline{y}}_j - \overline{\overline{\overline{y}}}\right)^2 + \sum_{i=1}^{p}\sum_{j=1}^{n}\sum_{k=1}^{r}(y_{ijk}-\bar{y}_{ij})^2 + r\sum_{i=1}^{p}\sum_{j=1}^{n}\left(\bar{y}_{ij} + \overline{\overline{\overline{y}}} - \overline{\overline{y}}_i - \overline{\overline{y}}_j\right)^2$$

We note above that:

$(\bar{\bar{y}})^2$ is scaled by rph ($\bar{\bar{y}}$ is based on rph observations);

$(\bar{y}_i - \bar{\bar{y}})^2$ is scaled by hr ($\bar{y}_i$ is based on rh observations);

$(\bar{y}_j - \bar{\bar{y}})^2$ is scaled by rp ($\bar{y}_j$ is based on rp observations);

$(\bar{y}_{ij} + \bar{\bar{y}} - \bar{y}_i - \bar{y}_j)^2$ is scaled by r ($\bar{y}_{ij}$ is based on r observations)

$S_{TOT} = S_{AVG} + S_P + S_H + S_{INT} + S_{PE}$

In our problem,

$S_{AVG} = 2007.04$	$\nu_{AVG} = 1$
$S_H = 293.42$	$\nu_H = h-1=3$
$S_P = 23.555$	$\nu_P = p-1=2$
$S_{PE} = 60.82$	$\nu_{pe} = hp(r-1) = 24$
$S_{TOT} = 2402.8$	$\nu_{TOT} = 36$

Obtain S_{INT} by Subtraction = 17.965

Alternatively, we can calculate the interaction term directly...

$$\text{Interaction Term} = \bar{y}_{ij} + \bar{\bar{y}} - \bar{y}_i - \bar{y}_j$$

	j=1,A	j=2,B	j=3,C	j=4,D
i=1	-.65833	0.275	-0.39167	0.775
i=2	0.76667	0.8	-0.36667	-1.2
i=3	-0.10833	-1.075	0.75833	0.425

$S_{INT} = 3\,((-.65833)^2 + (.275)^2 + ... + (.425)^2) = 17.965$
$\nu_{INT} = (h-1)\,(p-1) = (3)(2) = 6$

ANOVA Table

Source	S. of S.	D.of F	Mean Sqr.	F_{calc}
Average	2007.04	1	2007.04	791.99
HD Recipe	293.42	3	97.8067	38.595
Panelist	23.555	2	11.7775	4.647
Interaction	17.965	6	2.99417	1.182
Error	60.82	24	2.53417	
Total	2402.80	36		

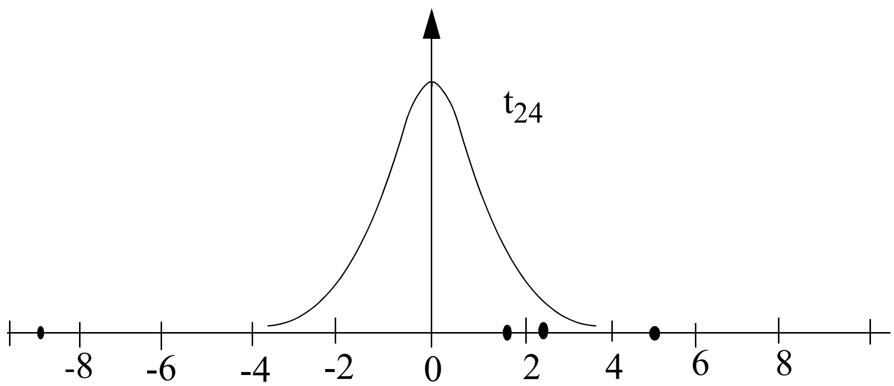
$F(1,24,.95) = 4.26$, $F_{TAB} < F_{CALC}$, the average is significant.
$F(3,24,.95) = 3.01$, $F_{TAB} < F_{CALC}$, recipe is significant.
$F(2,24,.95) = 3.40$, $F_{TAB} < F_{CALC}$, panelist is significant.
$F(6,24,.95) = 2.51$, $F_{TAB} > F_{CALC}$, interaction is not significant.

If interaction effect is significant it means that the recipe effect depends on the panelist. To confirm the ANOVA results, we can display the recipe and panelist means vs. the appropriate reference dist$^{\underline{n}}$

$s_Y^2 = 2.53417$, $s_Y = 1.59191$, $\nu=24$, $\overline{\overline{y}} = 7.4667$

Recipe

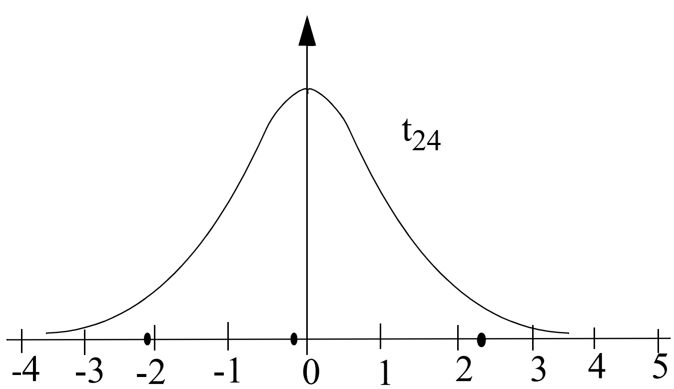
	A	B	C	D
$\overline{\overline{y}}_j$	8.7	10.16667	8.3333	2.6667
n	9	9	9	9
$s_{\overline{y}}$	0.5306	0.5306	0.5306	0.5306
t	2.32	5.09	1.63	-9.05



Hot Dog Recipe Means

Panelist

	1	2	3
$\bar{\bar{y}}_i$	6.525	8.5	7.375
n	12	12	12
$s_{\bar{y}}$	0.4595	0.4595	0.4595
t	-2.05	2.25	-0.20



Panelist Means Response

Residual Analysis

Express response with model:

$$y_{ijk} = \eta + P_i + H_j + INT_{ij} + \varepsilon_{ijk}$$

Mean (or intercept) is significant -- estimated as $\hat{\eta}$ = 7.4667

Panelists are significant -- estimates are

$$\hat{P}_i = \left(\bar{\bar{y}}_i - \bar{\bar{\bar{y}}}\right) = (-0.9416, 1.0333, -0.0916)$$

Recipes are significant -- estimates are

$$\hat{H}_j = \left(\bar{\bar{y}}_j - \bar{\bar{\bar{y}}}\right) = (1.233, 2.7, 0.8667, -4.8)$$

Interaction terms are not significant:

$\hat{y}_{ij} = \hat{\eta} + \hat{P}_i + \hat{H}_j$ is the model predicted response.

Residual error (difference between the actual and predicted responses) is:

$e_{ijk} = y_{ijk} - \hat{y}_{ij}$

$\boldsymbol{\hat{y}_{ij}}$

Panelist		Recipe			
		A	B	C	D
	1	7.7583	9.225	7.3917	1.725
	2	9.7333	11.200	9.3667	3.700
	3	8.6083	10.075	8.2417	2.575

Model Residuals

		Recipe			
Panelist		A	B	C	D
	1	-0.1583 -1.2583 -0.5583	2.175 -1.625 0.275	-1.0917 0.5083 -0.5917	0.975 1.375 -0.025
	2	-2.5333 3.8667 0.9667	1.7 1.2 -0.5	1.7333 -2.4667 -0.3667	-0.4 -1.8 -1.4
	3	-1.6083 1.5917 -0.30833	0.125 -1.975 -1.375	-1.4417 0.9583 2.7683	1.125 -0.375 0.525

The e's should be IIDN $(0,\sigma_y^2)$, i.e., Independently and Identically Distributed Normally with mean zero and unknown but fixed variance, σ_y^2. We need to check this!!

Suppose that the water content of the hot dogs stored in the smoke house decreases over time. Specific frankfurters are removed from the smoke house for cooking as needed to accomodate the experimental plan. Since the complete experiment may take several days to perform, the hot dogs extracted from the smoke house during the experiment will have decreasing levels of water content, and thus the hardness/texture of the hot dogs will increase throughout the experiment. Randomization will insure that this systematic trend is applied randomly to the different treatments (panelists and recipes) -- and meaningful conclusion can be drawn. Randomization also means that the significance tests we perform are approximately valid.

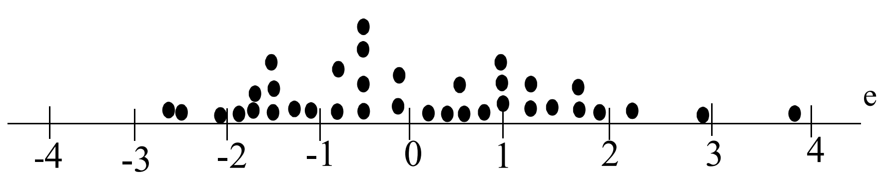
While randomization insures that we have no problems with analyzing and drawing conclusions from the experiment, it certainly would desirable to know if a smoke house effect did indeed exist, i.e., the time spent in the smoke house truly influences the hardness of the hot dogs. To identify the presence of such an effect, we may plot the model residuals as a function of time.

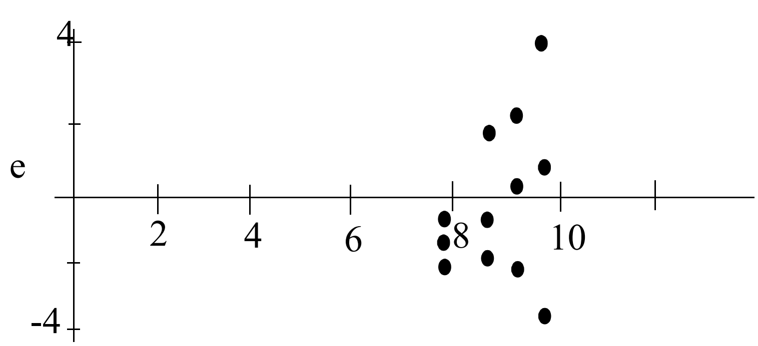
- Points to a source of variation - may be more carefully controlled in the future.
- More precise analysis in which the time trend is explicitly accounted for.

Types of Residual Plots

In general, we are interested in examining residuals to look for model inadequacies (e.g., problems with assumptions) or limitations. The philosophy here is that we're trying to learn as much as possible from the experiment. Discrepancies in the residuals point toward how we might run future experiments.

- Residuals vs. time
- Dot Diagram of all residuals
- Dot Diagram for each panelist
- Dot Diagram for each recipe
- Residuals vs. $\hat{y}$
- For replicated Data, $(y_{ijk} - \bar{y}_{ij})$ vs. $(\hat{y}$ or $\bar{y})$ (second plot)





From these residual plots, we look for structure, patterns, non-random behavior.

Variation described by model:

$$S_{model} = \sum_{i=1}^{p} \sum_{j=1}^{h} \sum_{k=1}^{r} \hat{y}_{ij}^2 = 2324.015$$

We note that $\Sigma_{model} = \Sigma_{AVG} + \Sigma_P + \Sigma_H$

$$S_{res} = \sum_{i=1}^{p} \sum_{j=1}^{h} \sum_{k=1}^{r} e_{ij}^2 = 78.785$$

We note that,

- $S_{res} = S_{pure\ error} + S_{INT}$
- $S_{model} + S_{res} = S_{TOT}$

To characterize the fit of the model,

$$R^2 = \frac{S_{MODEL}}{S_{TOT}} = \frac{2324.015}{2402.8} = 0.967$$

The model describes 96.7% of the variation in the data.

Sometimes instead of reporting this value, we subtract the variation due to the mean from the variation in the data and the variation in the mean. The resulting quantity describes the fraction of the variation described by the model.

$$R_{cor}^2 = \frac{S_{MODEL} - S_{AVG}}{S_{TOT} - S_{AVG}} = \frac{2324.015 - 2007.04}{2402.8 - 2007.04} = 0.801$$

In this case, 80% of the variation is described by the model.