12.0 FRACTIONAL FACTORIAL DESIGNS

(Updated Spring,2001)

We now need to look at a class of designs that are good for studying many variables in a relatively limited number of tests, fractional factorial designs.

Previously we saw that if we carefully selected levels for variables we could obtain a Latin square, or Graeco-Latin square design, we didn't need to look at all combinations of the variable levels. Fractional Factorial Designs, 2^{k-p} designs, are analogous to these designs.

Let's say we're thinking about a 2^3 full factorial design.

Test	I	1	2	3	12	13	23	123
1	+	-	-	-	+	+	+	-
2	+	+	-	-	-	-	+	+
3	+	-	+	-	-	+	-	+
4	+	+	+	-	+	-	-	-
5	+	-	-	+	+	-	-	+
6	+	+	-	+	-	+	-	-
7	+	-	+	+	-	-	+	-
8	+	+	+	+	+	+	+	+

We want to examine a 4th variable, but only have enough resources for 8 tests. We can introduce variable 4 thru interaction 123

Test	1	2	3	4
1	-	-	-	-
2	+	-	-	+

Test	1	2	3	4
3	-	+	-	+
4	+	+	-	-
5	-	-	+	+
6	+	-	+	-
7	-	+	+	-
8	+	+	+	+

Recipe Matrix: tells us how to run experiment

Test	I	1	2	3	4	12	13	14	23	24	34	123	124	134	234	1234
1	+	-	-	-	-	+	+	+	+	+	+	-	-	-	-	+
2	+	+	-	-	+	-	-	+	+	-	-	+	-	-	+	+
3	+	-	+	-	+	-	+	-	-	+	-	+	-	+	-	+
4	+	+	+	-	-	+	-	-	-	-	+	-	-	+	+	+
5	+	-	-	+	+	+	-	-	-	-	+	+	+	-	-	+
6	+	+	-	+	-	-	+	-	-	+	-	-	+	-	+	+
7	+	-	+	+	-	-	-	+	+	-	-	-	+	+	-	+
8	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

We note that the following columns are the same:

I&1234	3 & 124	23 & 14
1 & 234	12 & 34	123 & 4
2 & 134	13 & 24	

We have deliberately confounded Variable 4 and Interaction 123, in doing so we have also confounded I & 1234, 1 & 234, 2 & 134, etc. We have conducted a 2^{4-1} fractional factorial design 4 variables are examined in $2^{4-1} = 8$ tests.

Some Basic Ideas/Definitions

For 8 tests = $2^3 \rightarrow$ Base design Calculation Matrix is:

I	1	2	3	12	13	23	123
+	-	-	-	+	+	+	-
+	+	-	-	-	+	+	+
+	-	+	-	-	+	-	+
+	+	+	+	+	-	-	-
+	-	-	+	+	-	-	+
+	+	-	+	-	+	-	-
+	-	+	+	-	-	+	-
+	+	+	+	+	+	+	+

We introduced variable 4 thru int. 123. Symbolically, 4 = 123

- Any column multiplied by itself produces a column of all "+" signs, I.
- 4 • 4 = I
- Any column multiplied by I gives the original column:

I • 4 = 4
4• 4 • 4 = 4

Consider again how we introduced variable 4 into the design.

4 = 123

This is termed the generator for the design.

4 • 4 = 1234
and
I = 1234

is the alternative form for the generator.

In this case, this is the defining relationship for our 2^{4-1} design. It tells us the confounding/aliasing pattern for our design.

Question: Consider columns 123&4. How do we calculate E_{123} & E_4, in other words, estimates of μ_{123} & μ_4?

Since we only have 8 tests, we only can calculate 8 unique quantities. We calculate these quantities using the 8 columns in the base design calculation matrix (I,1,2,3,12,13,23,123)

$$\# = \frac{\sum_{i=1}^{Tests} Col_{12} \bullet y}{4} = l_{12}$$

These calculated values will be referred to as: $l_I, l_1, l_2, l_3, l_{12}, l_{13}, l_{23}, l_{123}$

We know what to calculate and how to do it, just like full factorials. How to interpret the results though?

We have observed that col. 123 = col. 4. When we calculate l_{123}, is this an estimate of μ_{123} or μ_4 or what? In fact, l_{123} estimates $\mu_{123} + \mu_4$. It is the effect of a linear combination of 123 & 4.

In our problem,

l_1 estimates μ_1+μ_{234} (shorthand notation for μ_1+μ_{234})
l_2 estimates μ_2+μ_{134}
l_3 estimates μ_3+μ_{124}
l_{12} estimates μ_{12}+μ_{34}
l_{13} estimates μ_{13}+μ_{24}
l_{23} estimates μ_{23}+μ_{14}
l_{123} estimates μ_{123}+μ_4

and
l_I estimates $I+\mu_{1234}/2$

Another example: Study 6 variable in 8 tests 2^{6-3} design. $2^{k-p} = 2^{m}$ tests
k = 6 variables to be studied
p = 3 variables to be introduced using interactions from base design
m = 3 number of variables in the base design

Base Design

Test	I	1	2	3	12	13	23	123
1	+	-	-	-	+	+	+	-
2	+	+	-	-	-	-	+	+
3	+	-	+	-	-	+	-	+
4	+	+	+	-	+	-	-	-
5	+	-	-	+	+	-	-	+
6	+	+	-	+	-	+	-	-
7	+	-	+	+	-	-	+	-
8	+	+	+	+	+	+	+	+

Introduce 4,5,6 using 3 of these interactions

Introduce variables 4,5, & 6 using interactions 12,13, & 23
Generators: 4 = 12, 5 = 13, 6 = 23 or I = 124, I = 135, I = 236

Recipe Matrix

Test	1	2	3	4	5	6
1	-	-	-	+	+	+
2	+	-	-	-	-	+
3	-	+	-	-	+	-
4	+	+	-	+	-	-
5	-	-	+	+	-	-
6	+	-	+	-	+	-
7	-	+	+	-	-	+
8	+	+	+	+	+	+

Defining Relationship

We have *p* generators
I = 124 = 135 =236

Consider their products 2 at a time

124 • 135 = 2345
124 • 236 = 1346
135 • 236 = 1256
Products 3 at a time
124 • 135 • 236 = 456

In general, all the way up to "products p at a time". So, Defining Relationship:

I = 124 = 135 = 236 = 2345 = 1346 = 1256 = 456

There are 2^p words = 8 in the defining relationship. "Word" = a string such as "236"

Design Resolution

"The length of the shortest word in the defining relation (excluding I) is termed the design resolution" For this example, Design Resolution = III.

What does Design Resolution mean?

- Des. Res. = III: Main effects are confounded with two factor interactions (1 with 2) and 3=1+2.
- Des. Res. = IV: Main effects are confounded with three factor interactions (1 with 3) and 4=1+3. It can also be two factor interactions are confounded with one another (2 with 2) and 4=2+2.
- Des. Res. = V: Main effects confounded with 4 factor c (1 + 4 = 5), or, two-factor interactions are confounded with three factor interactions (2 + 3 =5)
- Des. Res. = II: Main effects confounded with one another. This would be very, very bad.

Linear Combinations of Effects

Consider the columns in the base design calc. matrix: I,1,2,3,12,13,23,123. Multiply each of the columns above by each word in the defining relation.

l_I estimates

$$I+\frac{\mu_{124}}{2}+\frac{\mu_{135}}{2}+\frac{\mu_{236}}{2}+\frac{\mu_{2345}}{2}+\frac{\mu_{1346}}{2}+\frac{\mu_{1256}}{2}+\frac{\mu_{456}}{2}$$

l_1 est 1+24+35+1236+12345+346+256+1456

l_2 est 2+14+1235+36+345+12346+156+2456

l_3 est 3+1234+15+26+245+146+12356+3456

l_{12} est 12+4+235+136+1345+2346+56+12456

l_{13} est 13+234+5+126+1245+46+2356+13456

l_{23} est 23+134+125+6+45+1246+1356+23456

l_{123} est 123+34+25+16+145+246+356+123456

Note that each linear combination has 8 terms, the same as the number of words in the defining relation.

For 6 variables,
1 average
6 main effects
15 2-factor interactions
20 3-factor interactions
15 4-factor interactions
6 5-factor interactions
1 6-factor interaction

Total 64 effects

Conduct the 8 tests,
In std. order (24.5, 16.0, 16.0, 23.0, 25.0, 13.5, 17.0,24.0).
Under the assumption that 3 - factor and higher order interactions are negligible,

l_I = 20 estimates I

l_1 = -1.25 estimates $\mu_1 + \mu_{24} + \mu_{35}$

l_2 = 0.50 estimates $\mu_2 + \mu_{14} + \mu_{36}$

l_3 = 0.25 estimates $\mu_3 + \mu_{15} + \mu_{26}$

l_{12} = 8.75 estimates $\mu_{12} + \mu_4 + \mu_{56}$

l_{13} = -0.5 estimates $\mu_{13} + \mu_5 + \mu_{46}$

l_{23} = 1.25 estimates $\mu_{23} + \mu_6 + \mu_{45}$

l_{123} = 1.0 estimates $\mu_{34} + \mu_{25} + \mu_{16}$

Note that main effects are confounded with 2-factor interactions, and the design resolution is III.

Normal plot shows l_{12} important. Other μ's, may also be important but cancel one another out. If the sum " $\mu_{12} + \mu_4 + \mu_{56}$ "is important, it could be due to μ_{12} and/or μ_4 and/or

μ_{56}

Given no other experiment results, we must apply process knowledge to figure out what is important.

Fractional Fact Designs: Good for screening experiments. Find out important variables from a large list

- Want to develop/create a 2^{k-p} FFD. k-p=m
- Write out calculation matrix for 2^m full factorial design - base design
- Introduce the p new variables thru the interaction columns in the base design calculation matrix - p generators
- Write out the recipe matrix for the 2^{k-p} design
- Use the generators to develop defining relationship.
  - \# of words = 2^p
  - design resolution =length of shortest word in the defining relation.
- Write out the linear combination of effects that may be estimated - one linear combination for each col in the base design calculation matrix.
- Run expt. using recipe matrix
- Calculate l_I, l_1, l_2,... by multiplying each column in the base design calculation matrix by the column of responses and divide by the appropriate # of "+" signs.

More nomenclature:

2^7 full factorial = 128 tests

2^{7-1} frac. fact = 64 tests = 1/2 fraction

2^{7-2} frac. fact = 32 tests = 1/4 fraction

2^{7-3} frac. fact = 16 tests = 1/8 fraction

2^{7-4} frac. fact = 8 tests = 1/16 fraction

2^4 Full Factorial in Std. Order

Test	1	2	3	4
1	-	-	-	-
2	+	-	-	-
3	-	+	-	-
4	+	+	-	-
5	-	-	+	-
6	+	-	+	-
7	-	+	+	-
8	+	+	+	-
9	-	-	-	+
10	+	-	-	+
11	-	+	-	+
12	+	+	-	+
13	-	-	+	+
14	+	-	+	+
15	-	+	+	+
16	+	+	+	+

2^{4-1} design with 4=123

Test	1	2	3	4= 123
1	-	-	-	-
10	+	-	-	+
11	-	+	-	+
4	+	+	-	-
13	-	-	+	+
6	+	-	+	-
7	-	+	+	-
16	+	+	+	+

What about the remaining 8 tests?

Test	1	2	3	4	123
2	+	-	-	-	+
3	-	+	-	-	+
5	-	-	+	-	+
8	+	+	+	-	+
9	-	-	-	+	-
12	+	+	-	+	-
14	+	-	+	+	-
15	-	+	+	+	-

The generator for this 2^{4-1} design is 4 = -123. Thus, variable 4 is introduced by negating/switching/folding the signs on the 123 interaction column.

For 2^{4-1} design we can introduce variable 4 using 123 or -123. The generator is a member of
4 = $\pm$ 123 family of generators.

We might consider using the family of generators 4 = $\pm$ 12, but this is only of resolution III, while 4 = $\pm$ 123 is of resolution IV. We "Always" want designs with highest resolution.

Let's examine the linear combinations that are obtained from the defining relationships of our two 2^{4-1} designs with 4=$\pm$ 123

I = 1234	I = -1234
l_I est $I + \frac{\mu_{1234}}{2}$	l'_I est $I - \frac{\mu_{1234}}{2}$
l_1 est 1 +234	l'_1 est 1 - 234
l_2 est 2 +134	l'_2 est 2 - 134

l_3 est 3 +124	l'_3 est 3 - 124
l_{12} est 12 + 34	l'_{12} est 12 -34
l_{13} est 13 +24	l'_{13} est 13 -24
l_{23} est 23 + 14	l'_{23} est 23 - 14
l_{123} est 123 + 4	l'_{123} est 123 - 4

The results from the 2 FFD's may be combined to resolve effect confounding

$(l_I + l'_I)/2$ est I	$(l_I - l'_I)$ est 1234
$(l_1 + l'_1)/2$ est 1	$(l_1 - l'_1)/2$ est 234
$(l_2 + l'_2)/2$ est 2	$(l_2 - l'_2)/2$ est 134
$(l_3 + l'_3)/2$ est 3	$(l_3 - l'_3)/2$ est 124
$(l_{12} + l'_{12})/2$ est 12	$(l_{12} - l'_{12})/2$ est 34
$(l_{13} + l'_{13})/2$ est 13	$(l_{13} - l'_{13})/2$ est 24
$(l_{23} + l'_{23})/2$ est 23	$(l_{23} - l'_{23})/2$ est 14
$(l_{123} + l'_{123})/2$ est 123	$(l_{123} - l'_{123})/2$ est 4

Thus, if we were to run a 2^{4-1} FFD (4=123) followed up by a 2^{4-1} FFD (4=-123), we may calculate/recover all the information associated with a 2^4 full factorial