

Lecture #35

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Transportation Model

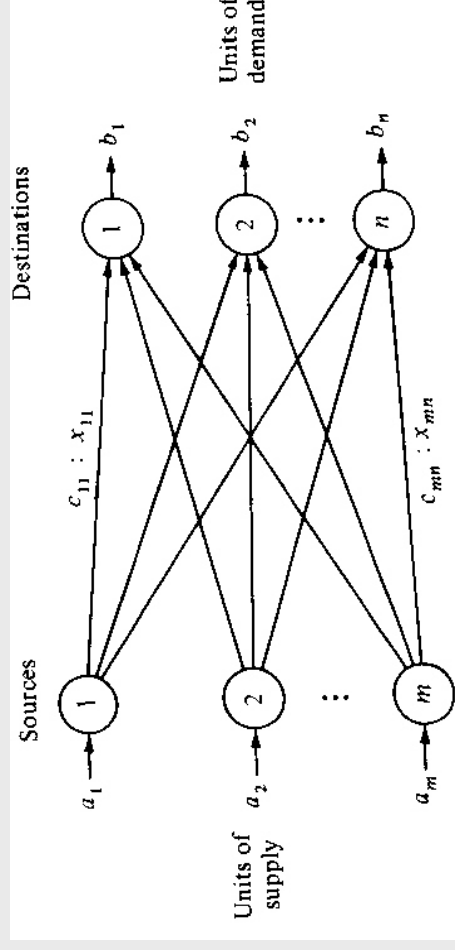
- ❖ This model deals with finding the a minimum cost plan for transporting a single commodity from a number of sources (e.g., factories) to a number of destinations (e.g., warehouses).
- ❖ The model can be extended in a direct manner to cover situations in areas related to inventory control, employment scheduling, personnel assignment, cash flow, etc.

Transportation Model

- ❖ The data of the model include:
 - ❑ Level of supply at each source and amount of demand at each destination
 - ❑ The unit transportation cost of the commodity from each source to each destination
- ❖ Since there is only one commodity, a destination can receive its demand from one or more sources. The objective is to determine the amount to be shipped from each source to each destination such that the total transportation cost is minimize.

Transportation Model

- ❖ The basic assumption of the model is that the transportation cost on a given route is directly proportional to the number of units transported.
- ❖ This figure depicts the transportation model as a network with m sources and n destinations.



Transportation Model

❖ The sources and destinations are represented by nodes. The arc joining a source and a destination represents the route through which the commodity is transported. The amount of supply at source i is a_i and the demand at destination j is b_j . The unit transportation cost between source i and destination j is c_{ij} .

Transportation Model

- ❖ x_{ij} represents the amount transported from source i to destination j ; then the LP model representing the transportation problem is given as:

$$\begin{aligned} & \text{subject to} \\ & \text{minimize } z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \\ & \sum_{j=1}^n x_{ij} \leq a_i, \quad i = 1, 2, \dots, m \\ & \sum_{i=1}^m x_{ij} \geq b_j, \quad j = 1, 2, \dots, n \\ & x_{ij} \geq 0 \quad \text{for all } i \text{ and } j \end{aligned}$$

Transportation Model

- ❖ A slightly tweaked model version requires that the total supply equal the total demand.
- ❖ When the total supply equals the total demand, the resulting formulation is called a balanced transportation model.
- ❖ It differs from the first one only in the fact that all inequality constraints are viewed as equalities:

$$\sum_{j=1}^n x_{ij} = a_i, \quad i = 1, 2, \dots, m$$
$$\sum_{i=1}^m x_{ij} = b_j, \quad j = 1, 2, \dots, n$$

Transportation Model Example

- ❖ The MG Auto Company has plants in L.A., Detroit, and New Orleans. Its major distribution centers are located in Denver and Miami. The capacities of the 3 plants during the next quarter are 1000, 1500, and 1200 cars. The quarterly demands at the two distribution centers are 2300 and 1400 cars. The train transportation cost per car per mile is approx. 8 cents.

Transportation Model Example

❖ Mileage chart:

	Denver	Miami
Los Angeles	1000	2690
Detroit	1250	1350
New Orleans	1275	850

❖ Costs (c_{ij}):

	Denver (1)	Miami (2)
Los Angeles (1)	80	215
Detroit (2)	100	108
New Orleans (3)	102	68

STD Transportation Model Example

- ❖ Using numeric codes to represent the plants and dist. centers, we let x_{ij} represent the number of cars transported from source i to destination j .
- ❖ Since the total supply ($=1000+1500+1200=3700$) happens to equal the total demand ($=2300+1400=3700$), the resulting transportation model is balanced.

Transportation Model Example

❖ Hence the following LP model presenting the problem has all equality constraints:

$$\text{minimize } z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32}$$

subject to

$$x_{11} + x_{12} = 1000$$

$$+ x_{21} + x_{22} = 1500$$

$$+ x_{31} + x_{32} = 1200$$

$$x_{11} + x_{21} + x_{31} = 2300$$

$$+ x_{12} + x_{22} + x_{32} = 1400$$

$$x_{ij} \geq 0 \quad \text{for all } i \text{ and } j$$

Transportation Model Example

❖ Another way for representing the transportation model is to use the transportation tableau – a matrix where the cost elements c_{ij} are given in the northeast corner

Table 5-1

		Destinations		Supply
		Denver (1)	Miami (2)	
Sources	Los Angeles (1)	80	215	1000
		x_{11}	x_{12}	
	Detroit (2)	100	108	1500
		x_{21}	x_{22}	
	New Orleans (3)	102	68	1200
		x_{31}	x_{32}	
Demand		2300	1400	

Transportation Model Example

❖ Solution to the model:

Table 5-1

		Destinations		Supply
		Denver (1)	Miami (2)	
Sources {	Los Angeles (1)	x_{11} 80	x_{12} 0	1000
	Detroit (2)	x_{21} 100	x_{22} 200	1500
	New Orleans (3)	x_{31} 102	x_{32} 1200	1200
Demand		2300	1400	
		1300	200	

The Assignment Model

- ❖ Consider the situation of assigning m jobs (or workers) to n machines. A job i ($i=1,2,\dots,m$) when assigned to machine j ($j=1,2,\dots,n$) incurs a cost c_{ij} . The objective is to assign the jobs to the machines (one job per machine) at the least total cost.
- ❖ The formulation of this problem may be regarded as a special case of the transportation model.

The Assignment Model

- ❖ Here jobs represent “sources” and machines represent “destinations.”
- ❖ The supply available at each source is 1; that is, $a_i=1$ for all i . Similarly, the demand required at each destination is 1; that is, $b_j=1$ for all j . The cost of “transporting” (assigning) job i to machine j is c_{ij} .

Job	Machine				
	1	2	...	n	
1	c_{11}	c_{12}	...	c_{1n}	1
2	c_{21}	c_{22}	...	c_{2n}	1
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
m	c_{m1}	c_{m2}	...	c_{mn}	1
	1	1	...	1	

The Assignment Model

- ❖ Before model can be solved by transportation technique, necessary to balance problem by adding fictitious jobs/machines, depending on whether $m < n$ or $m > n$.

$$x_{ij} = \begin{cases} 0 & \text{if the } j\text{th job is not assigned to the } i\text{th machine} \\ 1 & \text{if the } j\text{th job is assigned to the } i\text{th machine} \end{cases}$$

The model is thus given by:

$$\text{minimize } z = \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^n x_{ij} = 1, \quad i = 1, 2, \dots, n$$
$$\sum_{i=1}^n x_{ij} = 1, \quad j = 1, 2, \dots, n$$

$$x_{ij} = 0 \quad \text{or} \quad 1$$

The Assignment Model - Example

❖ Consider:

		Machine		
		1	2	3
Job	1	5 1	7 1	9 1
	2	14 1	10 1	12 1
	3	15 1	13 1	16 1

The Assignment Model - Example

- ❖ The optimal solution of the assignment model remains the same if a constant is added or subtracted to any row or column of the cost matrix.
- ❖ One can create a new c'_{ij} matrix with zero entries, and if these zero elements or a subset thereof constitute a feasible solution, this feasible solution is optimal -- because costs cannot be negative.
- ❖ In previous table zero elements are created by subtracting smallest element in each row (column) from the corresponding row (column).

The Assignment Model - Example

- ❖ If one considers the rows first, the new c'_{ij} matrix is:

$$\|c'_{ij}\| = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \hline 0 & 2 & 4 \\ 4 & 0 & 2 \\ 2 & 0 & 3 \end{array} \quad \begin{array}{l} p_1 = 5 \\ p_2 = 10 \\ p_3 = 13 \end{array}$$

- ❖ This last matrix can be made to include more zeros by subtracting $q_3=2$ from the third column:

$$\|c'_{ij}\| = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \hline \boxed{0} & 2 & 2 \\ 4 & 0 & \boxed{0} \\ 2 & \boxed{0} & 1 \end{array}$$

The Assignment Model - Example

- ❖ The squares in the last table give the feasible (and hence optimal) assignment (1,1), (2,3), and (3,2), costing $5+12+13=30$
- ❖ Unfortunately, it is not always possible to obtain a feasible assignment using this method. Further rules are thus required to find the optimal solution:

Assignment Model – Example 2

	1	2	3	4
1	1	4	6	3
2	9	7	10	9
3	4	5	11	7
4	8	7	8	5

❖ Now, carrying out the same initial steps, one gets:

	1	2	3	4
1	0	3	2	2
2	2	0	0	2
3	0	1	4	3
4	3	2	0	0

Assignment Model – Example 2

- ❖ Feasible assignment to the zero elements is not possible in this case.
- ❖ We may draw a minimum number of lines through some of the rows and columns such that all the zeros are crossed out.

	1	2	3	4
1	0	3	2	2
2	2	0	0	2
3	0	1	4	3
4	3	2	0	0

Assignment Model – Example 2

- ❖ The next step is to select the smallest uncrossed-out element, this element is subtracted from every uncrossed-out element and added to every element at the intersection of two lines

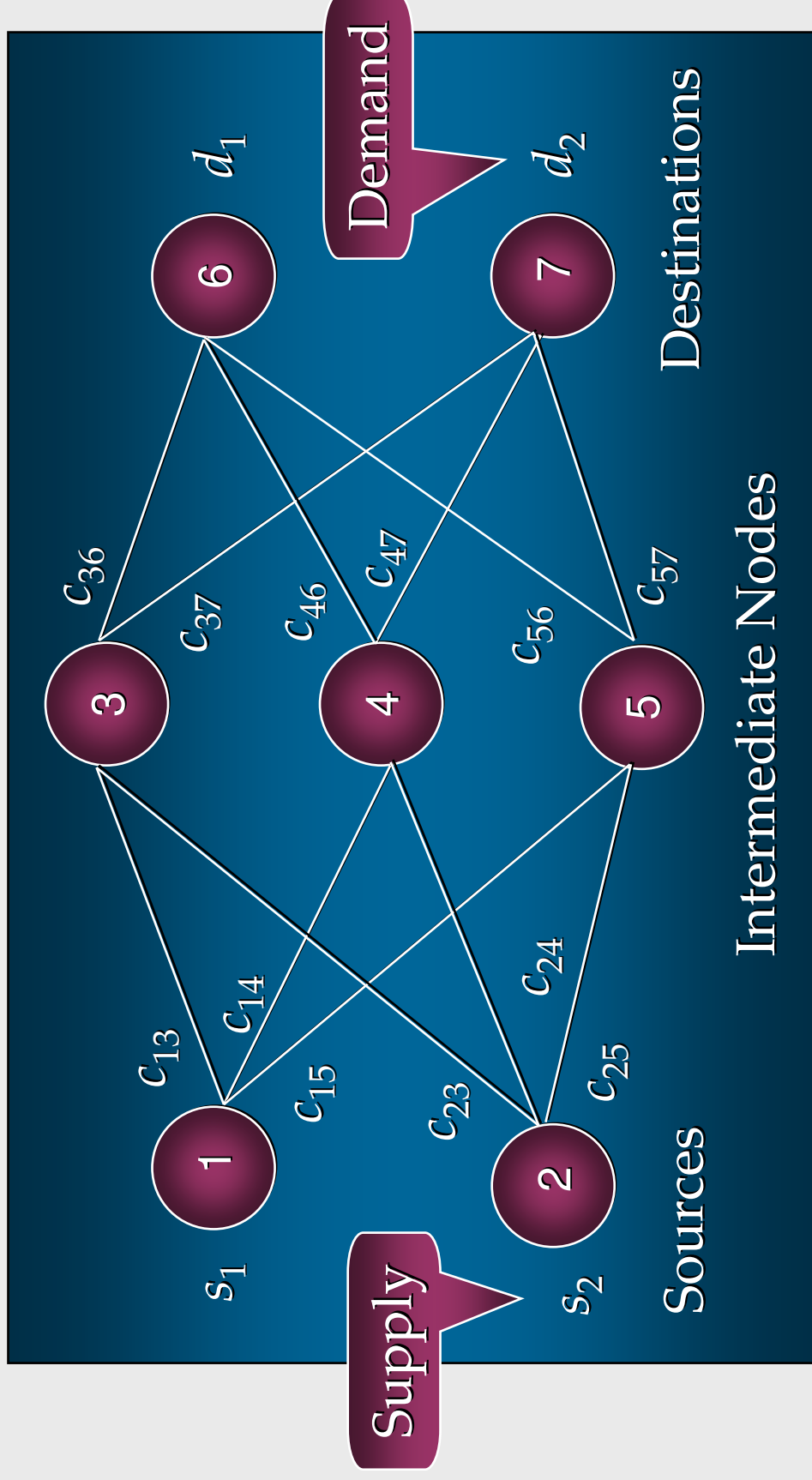
	1	2	3	4
1	0	2	1	1
2	3	0	0	2
3	0	0	3	2
4	4	2	0	0

- ❖ This gives the optimal assignment (1,1),(2,3),(3,2), and (4,4). The corresponding total cost is $1+10+5+5=21$

Transshipment Problem

- ❖ Transshipment problems are transportation problems in which a shipment may move through intermediate nodes (transshipment nodes) before reaching a particular destination node.
 - ❖ Transshipment problems can be converted to larger transportation problems and solved by a special transportation program.
 - ❖ Transshipment problems can also be solved by general purpose linear programming codes.
 - ❖ The network representation for a transshipment problem with two sources, three intermediate nodes, and two destinations is shown on the next slide.
- **Anderson/Sweeney/Williams: slide content – J. S. Loucks**

Transshipment Network



Transshipment Problem

❖ Linear Programming Formulation

x_{ij} represents the shipment from node i to node j

$$\text{Min } \sum_i \sum_j c_{ij} x_{ij}$$

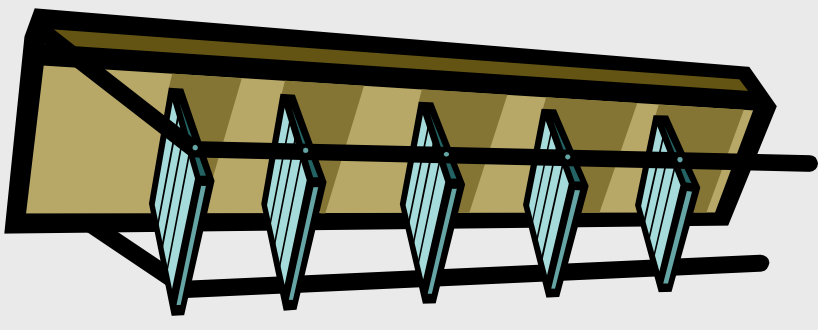
$$\text{s.t. } \sum_j x_{ij} \leq s_i \quad \text{for each origin } i$$

$$\sum_i x_{ik} - \sum_j x_{kj} = 0 \quad \text{for each intermediate node } k$$

$$\sum_i x_{ij} = d_j \quad \text{for each destination } j$$

$$x_{ij} \geq 0 \quad \text{for all } i \text{ and } j$$

Example: Zeron Shelving



- ❖ The Northside and Southside facilities of Zeron Industries supply three firms (Zrox, Hewes, Rockrite) with customized shelving for its offices. Zeron's two facilities both order shelving product from same two manufacturers, Arnold Manufacturers and Supershelf, Inc.
- ❖ Currently weekly demands by the users are 50 for Zrox, 60 for Hewes, and 40 for Rockrite. Both Arnold and Supershelf can supply at most 75 units to its customers.
- ❖ Additional data is shown on the next slide.

Example: Zeron Shelving

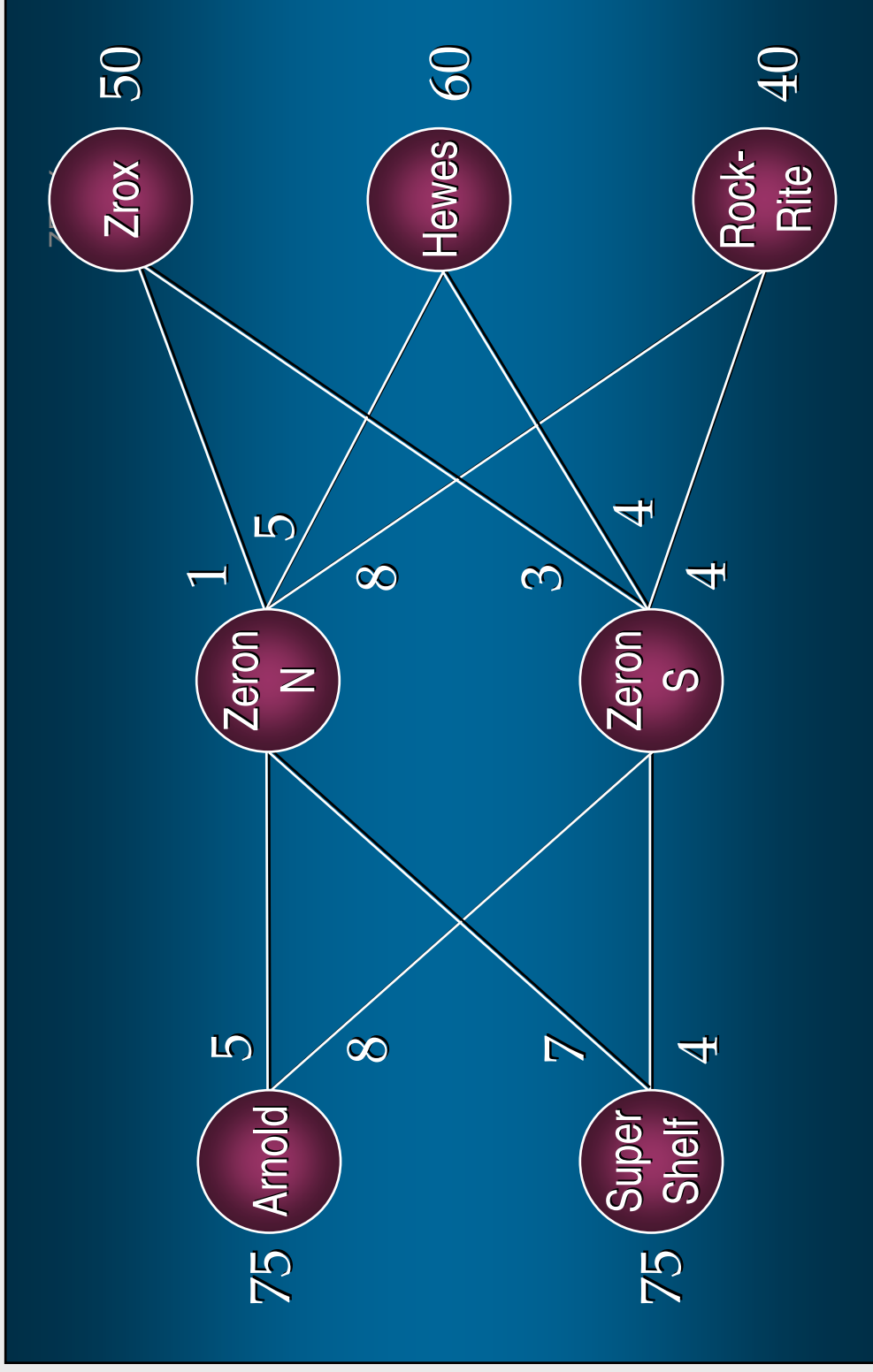
Because of long standing contracts based on past orders, unit costs from the manufacturers to the suppliers are:

	<u>Zeron N</u>	<u>Zeron S</u>
Arnold	5	8
Supershelf	7	4

The costs to install the shelving at the various locations are:

	<u>Zrox</u>	<u>Hewes</u>	<u>Rockrite</u>
Thomas	1	5	8
Washburn	3	4	4

Example: Zeron Shelving Network



Example: Zeron Shelving

- ❖ Linear Programming Formulation

- Decision Variables Defined

x_{ij} = amount shipped from manufacturer i to supplier j

x_{jk} = amount shipped from supplier j to customer k

where $i = 1$ (Arnold), 2 (Supershelf)

$j = 3$ (Zeron N), 4 (Zeron S)

$k = 5$ (Zrox), 6 (Hewes), 7 (Rockrite)

- Objective Function Defined

Minimize Overall Shipping Costs:

$$\begin{aligned} \text{Min} \quad & 5x_{13} + 8x_{14} + 7x_{23} + 4x_{24} + 1x_{35} + 5x_{36} + 8x_{37} \\ & + 3x_{45} + 4x_{46} + 4x_{47} \end{aligned}$$

Example: Zeron Shelving

❖ Constraints Defined

Amount Out of Arnold: $x_{13} + x_{14} \leq 75$

Amount Out of Supersheff: $x_{23} + x_{24} \leq 75$

Amount Through Zeron N: $x_{13} + x_{23} - x_{35} - x_{36} - x_{37} = 0$

Amount Through Zeron S: $x_{14} + x_{24} - x_{45} - x_{46} - x_{47} = 0$

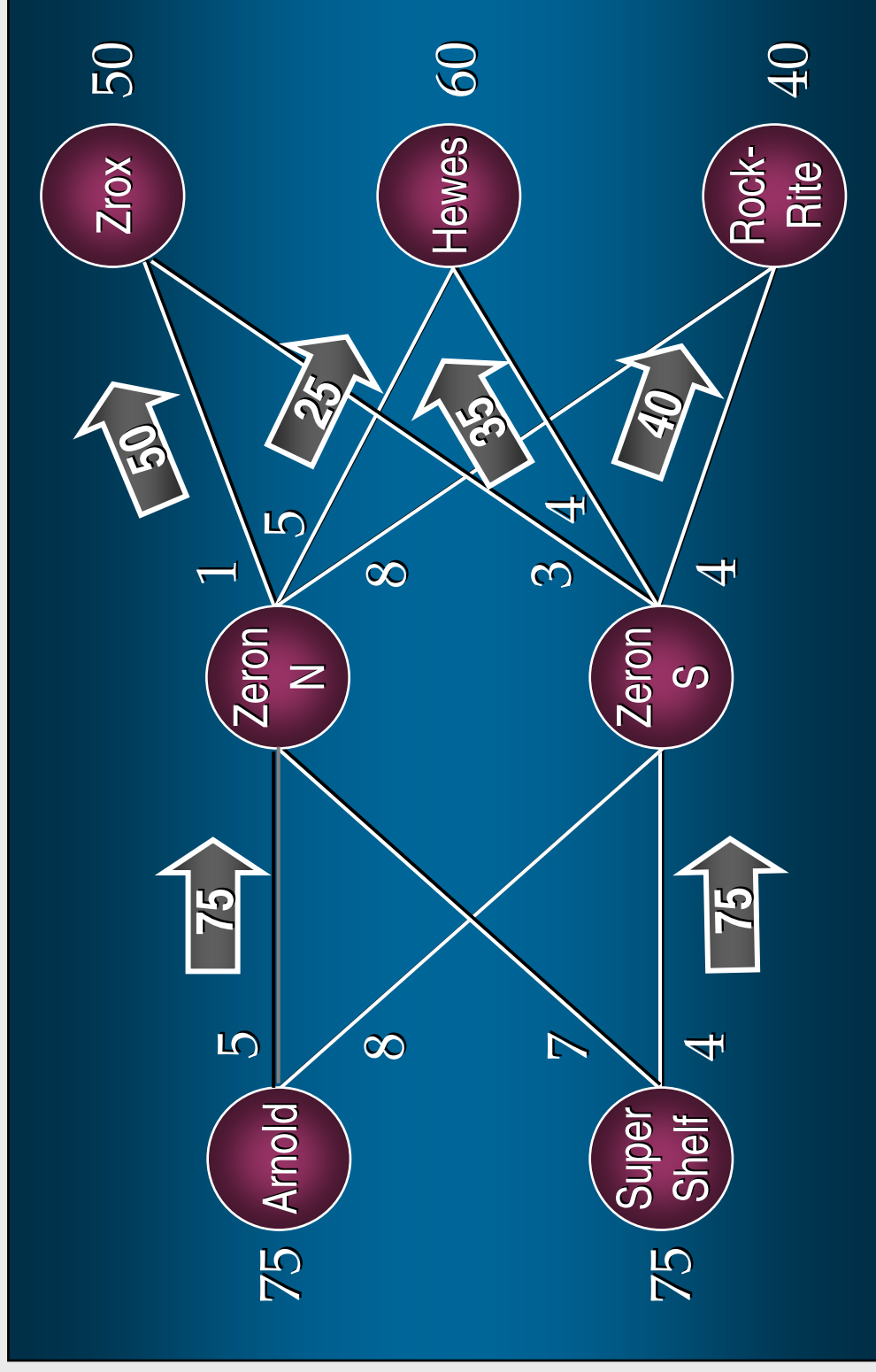
Amount Into Zrox: $x_{35} + x_{45} = 50$

Amount Into Hewes: $x_{36} + x_{46} = 60$

Amount Into Rockrite: $x_{37} + x_{47} = 40$

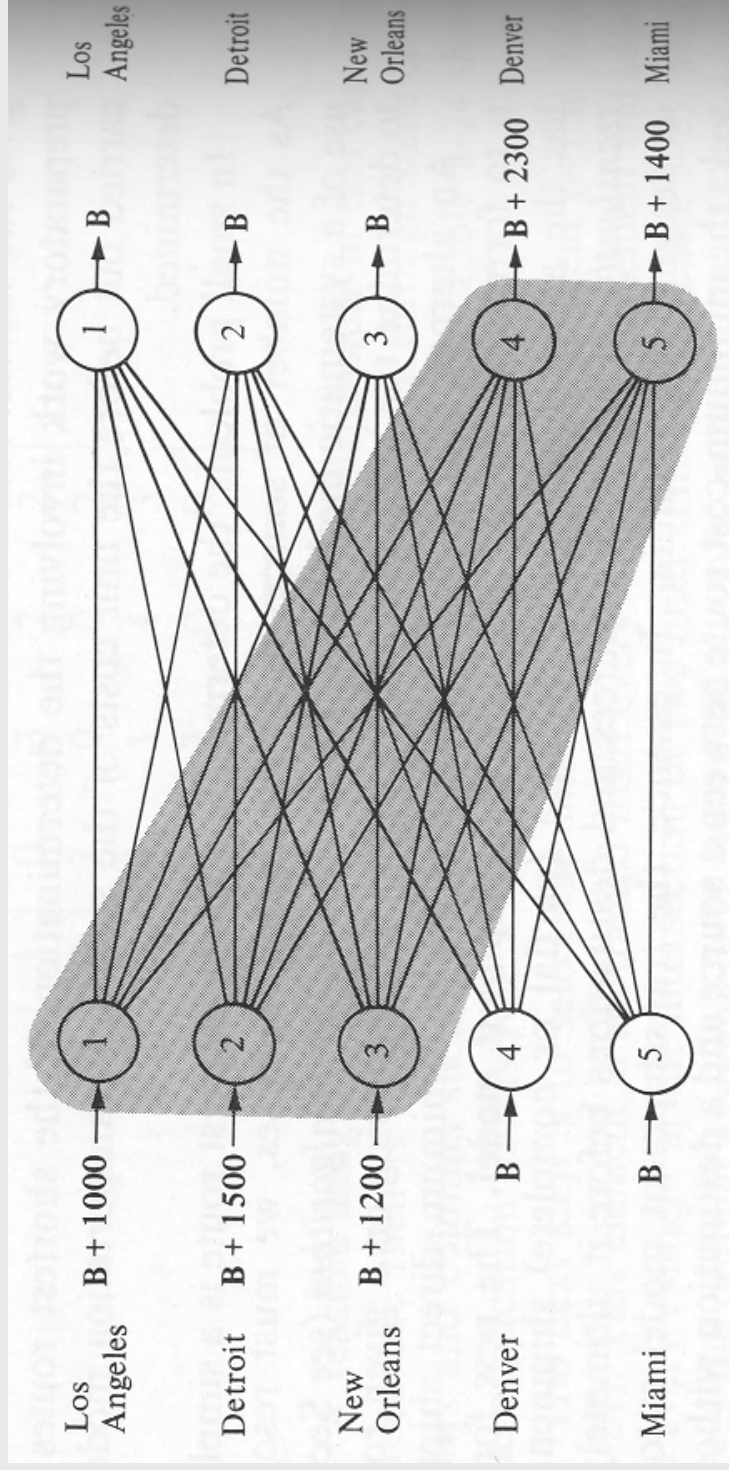
Non-negativity of Variables: $x_{ij} \geq 0$, for all i and j .

Example: Zeron Shelving -- Solution

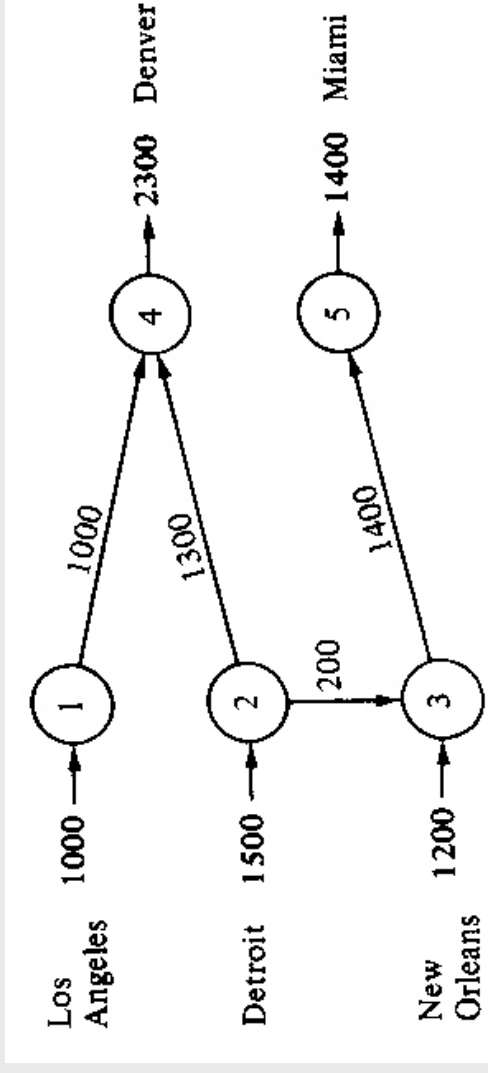


The Transshipment – MG Model

❖ In terms of the transshipment model, we have 5 sources and 5 destinations:



Optimal Solution



❖ Transshipping takes place between Detroit and the New Orleans plants, where 200 cars are shipped from Detroit to Miami via New Orleans. All other shipments are direct from the plants to the distribution centers.