The Magic of Math: 7th Grade

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1 Introduction

The Webb group works on theoretical and experimental aspects of laser physics and technology, and problems related to optomechanics, super-resolution imaging, imaging and sensing in scattering media, and neuroscience. All of this work relies heavily on mathematics. Consequently, the graduate and undergraduate students involved in this work pursue mathematical work daily that relates directly to the topics treated in the 7th grade math curriculum.

Four “stations” with themes that relate to research and education in electrical engineering are treated. Each is intended to be attended by about 5 students for 10 minutes. The leader for each station is a Ph.D. student at Purdue.

2 Station 1: CD Diffraction Grating – Laser Scatter from Objects

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Lasers are used in many systems we might use daily, including optical fiber communication – when your cell phone communicates with the tower, the signal from the tower likely moves through and optical fiber network within the United States and to other parts of the world. That information is conveyed through the fibers using laser light and a wavelength of about 1.55 µm (because the loss of silica fibers is low at that wavelength). Lasers are also used for machining, atmospheric measurements, and imaging. For instance, fluorescing proteins being studied under a microscope would be exciting using a laser.

Theory and Background: An optical disk has a set of pits that are used to store information such as data or music, and this information is read by scattering laser light from the disk as it spins. Incidentally, the repeating pattern of tracks in which the pits lie also forms what is known as a reflective diffraction grating. One can show with the Huygens–Fresnel Principle (where each point along a wavefront can be treated as its own point source) and far field approximations that the fields will interfere constructively in certain directions, and destructively in others.

One can show through some more involved mathematics that the directions of the intensity maxima are given by the well-known grating equation, written as

$$\sin(\theta_m) - \sin(\theta_i) = m\frac{\lambda}{d}, \tag{1}$$

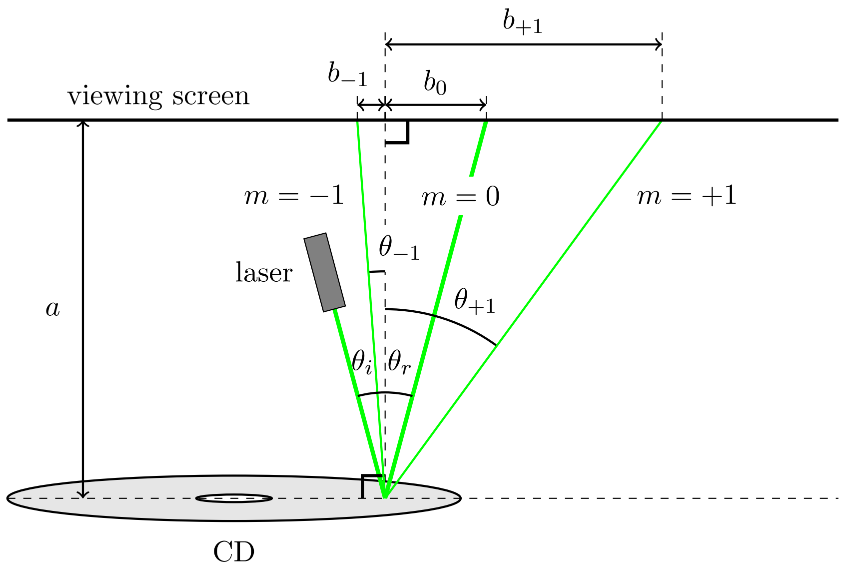

Figure 1. A depiction of the setup to be used to measure the quantities needed to calculate the optical disk track spacing, using a laser of known wavelength. Because of the right angles in the setup, all quantities of interest can be found by measuring only a, b_0, and $b_{\pm 1}$. θ_r is the usual angle of reflection, $\theta_r = \theta_i$, which is also considered the $m = 0$ diffraction order angle.

where θ_i is the angle of incidence (positive θ_i being counterclockwise from the surface normal), λ is the wavelength of the illuminating radiation, d is the grating period, $m = 0, \pm 1, \pm 2, \ldots$ is an integer corresponding to the "diffraction order," and θ_m is the angle of maximum intensity of diffraction order m (referenced to the surface normal, positive θ_m being clockwise). Which sine function is subtracted from which on the left hand side of the equation is merely a matter of convention, since m can be either positive or negative. Here we choose the sign convention resulting in positive m orders being further from the surface normal than the zeroth-order.

Although the physics and math that result in this formula are far beyond what would be expected in a seventh-grade math class, the experimental setup depicted in Figure 1 allows for the determination of all quantities on the left-hand side of equation (1) using only a rectangular surface, a ruler, and the Pythagorean theorem. With a known laser wavelength, this allows for the determination of track spacing (or *pitch*) on an optical disk, or laser wavelength if the track spacing is known, with only a few simple computations. The activity area would be set up by a graduate student before the arrival of any seventh-grade students. The graduate student would also be in control of any lasers.

Activity Setup and Measurements:

1. Place some sort of straight reference line (such as the edge of a piece of tape) along the surface of the table, exactly perpendicular to the two edges on which it ends. (If a protractor or square is available, this can be used to ensure perpendicularity.) This will act as the reference for the surface normal; i.e. the dashed vertical line near the middle of Figure 1. Measure the reference line's length; this is distance a in Figure 1.

2. Attach the optical disk to a side of the table with which the reference line is perpendicular. Attach it such that the reference line meets the grating portion of the disk, and such that the middle of the disk is slightly above the table edge, so that the laser meets the grating at the

appropriate height.

3. Attach a viewing screen to the other edge of the table, such that the distance of intensity peaks from the reference line can be easily measured. This could be a piece of paper that could be marked, or a meter (or yard) stick from which measurements could be read directly.

4. With the laser parallel with the table top, shine it such that it impinges on the optical disk surface at the point where the reference line meets the disk surface. No specific angle is required, as long as at least the zeroth and positive first diffraction orders' intensity maxima are visible on the viewing screen (the negative first order may be oriented on the laser side of the surface normal). Mark the distances of the zeroth and first diffraction orders' intensity maxima from the reference line (distances b_0 and b_{+1}, respectively).

Students' Calculations: To avoid introducing trigonometric functions, the left side of equation (1) can be rewritten with the geometric definition of the sine of an angle for $m = +1$, as

$$\frac{b_{+1}}{c_{+1}} - \frac{b_0}{c_0} = \frac{\lambda}{d}, \tag{2}$$

where c_m is the hypotenuse length of the right triangle formed by the reference line, the viewing screen, and the optical path of diffraction order m. Since this is the hypotenuse of a right triangle, $c_m = \sqrt{a^2 + b_m^2}$.

When the laser wavelength λ is known, the computed ratios of triangle sides can be used to compute d, the center-to-center pitch of the tracks on the optical disk (for CDs, this is nominally 1.6 µm [1]). After d has been calculated from the measurements, a different laser of "unknown" wavelength can be used, and the same measurements can be taken again, this time using them to find λ, effectively using a CD as a spectrometer. Some common wavelength options would be 650 nm, 532 nm, and 405 nm.

The students would thus perform the following mathematical steps, which could be outlined on a worksheet with a simplified diagram to streamline the process:

Computational Steps:

1. Measure the lengths b_0 and b_{+1} on the viewing screen.

2. Using the known length a, compute the lengths of the hypotenuses c_0 and c_{+1} using the Pythagorean theorem, $c_m = \sqrt{a^2 + b_m^2}$.

3. Subtract the ratios of the measured and computed quantities: $(b_{+1}/c_{+1}) - (b_0/c_0)$.

4. The final step depends on which quantity (λ or d) is known, and which is being found:

   (a) **If finding track pitch (d) from known wavelength (λ)**: Divide the known laser wavelength λ (in µm) by the difference found in Step 3 above to find the track pitch d (in µm).

   (b) **If finding wavelength (λ) from known track pitch (d)**: Multiply the difference obtained in Step 3 by the track pitch d (in µm) to obtain the wavelength λ (in µm). This could be either the nominal track pitch (1.6 µm), or the one found with a laser of known wavelength.

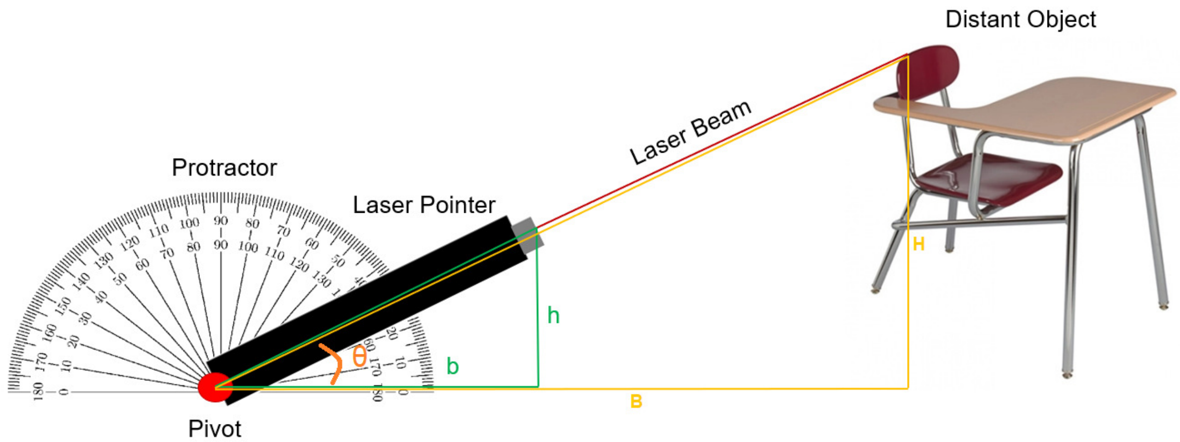

Figure 2. A depiction of the setup to be used to measure the height of a distant object with the inclinometer. Students will measure b_1, b_2, h_1 and θ

3 Station 2: Inclinometer for Measuring Height – Laser Ranging and Motion Sensing

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Lasers are often used in industry and academic research to measure distances or the motion of objects. These measurements are generally based on time of flight or geometric measurements. Most commercial laser distance sensors operate based on time of flight measurements. The sensor sends out a laser pulse which reflects off of a target object and returns to the sensor. The sensor then records the time it took for the pulse to return, and computes the object distance based on the speed of light. There are many possible laser sensing configurations for measuring distance which are based on geometry as well. One of the Webb group's active experiments uses this type of sensor. In this experiment, a laser beam is shined at an angle on a membrane. As the membrane moves back and forth, the position where the laser beam reflects off of the membrane moves left or right, causing the beam spot on a detector to move horizontally as well. This allows the position of the membrane to be tracked with a high level of accuracy.

By mounting a laser pointer to a protractor, a simple inclinometer can be constructed which can measure the height of a distant object based on the same principle. In this activity, students will use this type of inclinometer to calculate the height of various objects and explore some basic principles of geometry and trigonometry. A diagram of the experimental setup students will work with is shown in figure 2.

Students' Calculations: The mathematics of the inclinometer will initially be explained to students based on the properties of similar triangles. The triangle formed by the top and bottom of the measured object and the pivot of the laser pointer (shown in yellow) is similar to the triangle formed by the tip of the laser pointer, the ground, and the pivot (shown in green). As a result, the ratio of the height to the base of the two triangles will be the same. Thus, the height of the distant object can be determined using

$$h_2 = b_2 \, \frac{h_1}{b_1}. \tag{3}$$

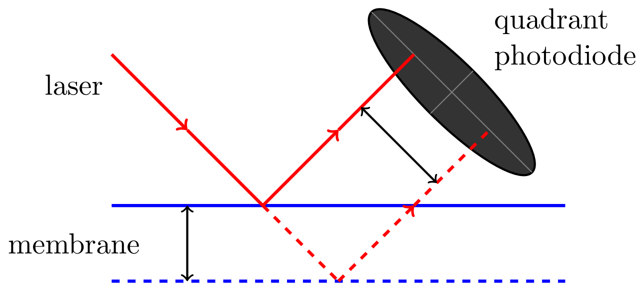

Figure 3. An illustration depicting the intended principle of operation used in membrane motion sensing in a Webb group experiment. The incident laser (red) impinges on the membrane obliquely, reflecting to the quadrant photodiode (black). As the membrane moves (to the dotted blue), remaining parallel to its original orientation, the reflected beam moves laterally (to the dotted red). This lateral motion is detected by the quadrant photodiode. While broadly similar to the inclinometer in that a laser is used to measure a distance, its principle of operation differs in that the lateral displacement of a beam is measured. In contrast, the inclinometer measures the angle a beam makes with some reference line.

Students will be given a worksheet showing the geometry involved and outlining the measurements that they need to take and the computational steps required to measure the height of the distant object. Students will then be asked to make the same calculation using the angle measured on the protractor and trigonometric functions. Using this approach, the height of the object can be expressed as

$$h_2 = b_2 \ \tan\theta. \tag{4}$$

The instructor will explain that these functions act as a sort of "lookup table" for the ratios of the side lengths of right triangles, and how this is a useful tool in science and engineering.

Experimental Procedure:

1. Place the inclinometer on the ground or a suitable flat surface.
2. Measure a position 3 meters away from the pivot of the inclinometer using a measuring tape and mark it with a piece of tape.
3. Place a large object at the marked position (a backpack, a desk, a lamp etc.).
4. Rotate the laser pointer on the pivot until the beam touches the top of the object.
5. Measure the height of the tip of the laser pointer from the ground and the base length of the triangle formed by the laser pointer and pivot using a ruler.
6. Measure the angle between the laser pointer and the ground.
7. Compute the height of the distant object using the similar triangle formula.
8. Compute the height of the distant object using the trigonometric formula.
9. Measure the height of the object using a meter stick or measuring tape and compare to the computed result.
10. Repeat the procedure with various objects.

4 Station 3: Estimating π – Understanding Physical System Models

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The parameter π is used in countless situations and in virtually all aspects of our research. Its properties and their impact is truly remarkable. It is the Greek letter for the "p" sound and the Greeks were very interested in the properties of circles. This might have been of practical importance in building the columns in the Parthenon in Athens, or perhaps they were just curious! Some people celebrate Pi Day, March 14 (3/14), and this was Albert Einstein's birth date (March 14, 1879).

The value for π is known to many decimal places and, to five decimal places has the value 3.14159. If r is the radius of a circle, then the circumference is $2\pi r$ and the area is πr^2. For a sphere of radius r, the surface area is $4\pi r^2$ and the volume is $4\pi r^3/3$.

While π can be estimated using many different procedures, we consider two of these that are summarized in the 7th grade math textbook [2].

Method 1: With a tape measure, measure the circumference of three different objects with a circular cross section (more or less, these would be circular cylinders). Then measure the diameter $d = 2r$, with r the radius. Tabulate each of these measurements. We no estimate π as Circumference/d. How close to 3.14159 did your result become and why might the results differ? From these measurements, how might you obtain a better estimate for π?

Method 2: Again, we must find the circumference of a circle. If it has unit radius ($r = 1$), then the circumference should be numerically equal to 2π. We use inscribed (inside the circle) and circumscribed (outside the circle) polygons [3, 4]. The principle is that a polygon drawn around the circle will have a total length greater than that drawn inside, where the vertices touch the circle. As the number of sides of the polygon increase, the inscribed and circumscribes results approach the circumference of the circle. We can use this to estimate π. This is an activity in the 7th grade math book [2].

Step (i): Draw squares, hexagons, and octagons inside and outside a circle. For the inside polygons, the vertices touch the circle and for the outside polygons it is the sides of the polygon that touch. In the first case, that of the square, the outside square has side equal to the diameter of the circle and the inside one has a diagonal equal to the diameter of the circle.

Step (ii): Measure the perimeter (length) of the outside and inside polygons and record those results. Then calculate the average (sum each and divide by 2). This becomes an estimate of the circumference of the circle (one result being slightly larger and the other a little less).

Step (iii): Divide each of the average polygon perimeters by the diameter of the circle to estimate π. What do you notice at the number of sides on the polygon increase?

Illustrative Application: Consider a simple model for the antenna in your cell phone. Let us say that your cell phone radiates a total of P_c W (the unit for power is Watt, with the symbol W) when you speak with a friend. An electromagnetic wave is radiated by the antenna in the phone to a cell phone tower antenna somewhere nearby, and then the signal moves through optical fiber or other transmission systems, and is transmitted to your friends phone, even if they are sitting nearby. Consider the simple (and slightly incorrect picture) that your phone radiates the same signal in all directions. This would mean that the power density (W/m^2) at the cell phone tower r m (m is meters) is

$$S_r = \frac{P_r}{4\pi r^2} \quad (\mathrm{W/m^2}), \tag{5}$$

because we think of the cell phone tower being on the surface of a sphere r m from your phone. We can think of the antenna on the cell phone tower as having an effective area A_e m^2. This allows us to estimate how much power comes out of the antenna on the cell phone tower and into the network as

$$P_d = S_r A_e \ \ \text{(W)}. \tag{6}$$

Knowing the lowest value of P_d would be important in designing the cell phone communication system, because there is background noise and the received signal needs to be larger than the noise and hence considered in the design of this communication system.

5 Station 4: Monty Hall Problem – Statistics and Probability Theory

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Probability is a unique branch of math because, in a way, it's mostly a lie. Sure, if you flip a coin many times in a row it will probably give about 50% heads and 50% tails. If you talk to physicists though, they'll tell you that, if you flip a coin *exactly* the same way twice in a row (making sure even the smallest details are the same), the physics says the coin should travel in the same way both times. So who is right? Is the world around us really completely predictable, or is it random?

The truth is that, while many things in the world are not fundamentally random, probability theory can make it a lot easier to think about certain things. It is true that, at the very small scale in physics, some things really are fundamentally random. Even outside this small scale, though, any experimental measurements made in a lab will have some amount of "noise" that is most easily modeled as random. In a simple example, the Monty Hall problem makes no sense when thinking about it intuitively, but is easily explained using probability theory.

Monty Hall: The Monty Hall problem comes from a game show. Before the contestant are three doors: behind one of them is a car, and behind the other two are goats. The contestant wins if they correctly guess the door with the car. The game begins with the contestant randomly choosing a door, let's call it Door A. After this comes the twist. Monty (who knows where the car is) opens one of the doors that has a goat behind it. Let's call that Door B. The contestant now knows that the car must be behind either Door A or Door C. Finally, Monty asks a devious question: "Would you like to switch your guess to Door C?"

Well, should you switch? Does it matter? At first, it may seem like it may not matter, because it feels like another 50-50 chance of winning either way. However, if you play the game many times in a row, you'll find that you win more often if you do switch to Door C. Why is this the case? This is the Monty Hall problem.

Exercise: Students pair up and take turns playing the Monty Hall game using cards (one face card and two non-face cards). Each pair decides which one of them will use the "stay" strategy, and who will use the "switch" strategy. As they play, they keep a tally record of how many wins and losses the two strategies result in. At the very end, all the counts are collected together to see which strategy ended up having a better win rate.

Explanation: Figure 4 shows a table of all possible outcomes when playing the Monty Hall game. A common misconception is that, after Monty reveals one of the goat doors, the probability changes to 50% that you picked the correct door. However, this is not the case. Right at the beginning, you had a 1/3 chance of choosing the door with the car, and this doesn't change even after Monty gives you a bit of information later.

Statology.org				
Contestant Picks This Door	Prize is Behind This Door	Monty Shows that Prize is Not Behind This Door	Result if Contestant Stays	Result if Contestant Switches
1	1	2 or 3	Win	Lose
1	2	3	Lose	Win
1	3	2	Lose	Win
2	1	3	Lose	Win
2	2	1 or 3	Win	Lose
2	3	1	Lose	Win
3	1	2	Lose	Win
3	2	1	Lose	Win
3	3	1 or 2	Win	Lose
		Win %	33%	66%

Figure 4. Why is switching doors the best strategy for the Monty Hall problem? This table shows all the different possible outcomes, all of which are equally likely. Try counting the number of winning outcomes for each strategy.

References

[1] "Audio recording - Compact disc digital audio system," International Standard, International Electrotechnical Commission, Geneva, CH (1999), IEC 60908:1999.

[2] R. Larson and L. Boswell, *Big Ideas Math: A common core curriculum* (Big Ideas Learning, 2014).

[3] A. Benjamin, *The Magic of Math: Solving for x and Figuring Out Why* (Basic Books, 2015).

[4] S. H. Strogatz, *The Joy of x: a Guided Tour of Math, from One to Infinity* (Houghton Mifflin Harcourt, 2012).