LMI CONTROL DESIGN FOR NONLINEAR VAPOR COMPRESSION CYCLE SYSTEMS

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ABSTRACT
To effectively control vapor compression cycle (VCC) systems whose dynamics are highly nonlinear, it is necessary to develop plant models and control laws for different operating regions. This paper presents a first-principles modeling framework that captures four operation modes over the operating envelope to construct an invariant-order switched system. To synthesize a multi-input multi-output (MIMO) control system, the Linear Quadratic Regulator (LQR) technique is framed as a control optimization problem with Linear Matrix Inequality (LMI) constraints which can be simultaneously solved for the set of considered linear systems. Stability and performance characteristics of the controlled system are guaranteed using a common quadratic Lyapunov function. Simulation results in a case study show that the LMI-based controller can maintain system operation at optimal set-points with mode switching over a wide operating envelope.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>h</td>
<td>Specific Enthalpy</td>
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<tr>
<td>m</td>
<td>Mass flow rate</td>
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<tr>
<td>P</td>
<td>Pressure</td>
</tr>
<tr>
<td>Q</td>
<td>Heat transfer rate</td>
</tr>
<tr>
<td>T</td>
<td>Temperature</td>
</tr>
<tr>
<td>W</td>
<td>Power</td>
</tr>
<tr>
<td>ω</td>
<td>Compressor speed</td>
</tr>
<tr>
<td>η</td>
<td>Isentropic efficiency</td>
</tr>
<tr>
<td>Ψ</td>
<td>Mean void fraction</td>
</tr>
<tr>
<td>ζ</td>
<td>Fraction of heat exchanger length covered by zone</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Subscript</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>Condenser</td>
</tr>
<tr>
<td>e</td>
<td>Evaporator</td>
</tr>
<tr>
<td>in</td>
<td>Inlet</td>
</tr>
<tr>
<td>out</td>
<td>Outlet</td>
</tr>
<tr>
<td>ref</td>
<td>Refrigerant</td>
</tr>
<tr>
<td>w</td>
<td>Heat exchanger wall</td>
</tr>
<tr>
<td>c1, c2, c3</td>
<td>Superheated, two-phase, subcooled zone in condenser</td>
</tr>
<tr>
<td>e1, e2</td>
<td>Two-phase, superheated zone in evaporator</td>
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INTRODUCTION
The primary goal of any air conditioning or refrigeration (AC/R) application is to move energy from one location to another. An ideal vapor compression cycle (VCC) system, as shown in Fig. 1, is a thermodynamic system driven by the phase characteristics of the refrigerant that is flowing through it. VCC systems transfer heat by means of repeating phase transitions in the refrigerant which can lead to the changing of the fluid characteristics in each of the heat exchangers (condenser and evaporator). The mixture of liquid and vapor refrigerant distributed in the heat exchangers makes the phase transition process highly nonlinear and complex [1-3]. Therefore, it is necessary to develop control strategies which account for system nonlinearities and maintain desirable performance over the entire operating envelope.

Advanced control of VCC systems in a variety of AC/R applications has been widely studied in the literature [4-8]. In particular, multi-input multi-output (MIMO) control design is attractive due to the strong coupling in VCC systems. He et al. [4] proposed a Linear Quadratic Gaussian (LQG) MIMO controller with gain scheduling to adapt to changing operating conditions. Rasmussen and Alleyne [5] presented a Youla parameterization framework for the generation of a local model and local controller network (LMN/LCN) and demonstrated the effectiveness of MIMO gain scheduling control strategies on performance regulation of the nonlinear system. A LQG controller based on a first-principles simulation model was developed in [8], and the controller performance was evaluated for operating conditions 30% away from the nominal operating point [9].
FIGURE 1. SCHEMATIC OF AN IDEAL VCC SYSTEM.

Although it has been shown that VCC systems exhibit nonlinear behavior across their entire operating envelope, a particular heat exchanger formulation with a fixed number of fluid zones is normally assumed for VCC system control design in the open literature. It has also been identified that highly nonlinear dynamics evolve when the system loses its condenser subcooled zone or evaporator superheated zone [3]. These types of refrigerant transitions can occur due to varying system boundary conditions. This paper goes beyond previous efforts to develop a control framework that is capable of handling the transitions inside the fluid zones. Motivated by recent work on dynamic modeling of refrigerant phase transitions [10], this paper presents a switched model network, thereby enabling model-based MIMO control design for the nonlinear VCC system. The VCC system can be represented by a set of four different operating modes that span the normal operation envelope. The key contribution of this work is the formulation of an invariant-order first-principles switched model system amenable to switched system controller design across the normal operating envelope of a typical VCC system.

The rest of this paper is organized as follows. Section 2 introduces the switched model network in terms of the different modes of refrigerant distribution during operation. A first-principles dynamic model for each operation mode is described in Section 3 along with the linearized model validation. Pseudo-state dynamic equations are used to maintain the same order in each mode of the switched model network. Section 4 presents a Linear Matrix Inequality (LMI) representation of the Linear Quadratic Regulator (LQR) control design. A model-based switching control scheme with set-point optimization is presented in Section 5. Simulation results demonstrate the advantages of the switched model framework on nonlinear VCC system control across its operating envelope. A conclusion section summarizes the main point of this paper.

FIGURE 2. FOUR OPERATION MODES OF THE VCC SYSTEM.

SYSTEM OPERATION

For this work, a moving-boundary model formulation [2] is utilized to describe all heat exchanger dynamics. As discussed in [10, 11], the refrigerant phase transitions in the condenser switch back and forth between 3-zone (superheated, two-phase and...
subcooled) and 2-zone (superheated and two-phase) model representations under different operating conditions. There also exist two different refrigerant distribution configurations in the evaporator model, which are 2-zone (two-phase and superheated) and 1-zone (two-phase) model representations. Therefore, the VCC system consists of four different operation modes across its operating envelope as depicted in Fig. 2. Exogenous signals, such as a desired reference trajectory or system disturbances, and/or endogenous scheduling variables, such as measured pressure or temperature signals [5] can drive the VCC system to switch between different operation modes. The switched operation mode framework is summarized in Fig. 3.

![FIGURE 3. SCHEMATIC OF THE SWITCHED OPERATION MODE NETWORK.](image)

### SYSTEM MODELING AND LINEARIZATION

As mentioned earlier, the VCC system dynamics are highly nonlinear. In order to apply linear control techniques on the VCC system, a linearized model of the system dynamics is needed. In the case of the switched model network, the nonlinear dynamic model for each individual operation mode is linearized. The interested reader is referred to [10, 11] for the nonlinear model derivations and to [12] for the linearization procedure.

The dynamic states of the switched model are described in Eq. (1). First-order pseudo-state equations [1] are added in each operation mode to ensure the continuity of states and maintain a 13th-order switched model of the system.

\[
x = [\xi, P, h_3, T_{ev}, T_{co}, T_{ai}, P, h_2, T_{ei}, T_{ae}, \gamma, \tau] \quad (1)
\]

The state-space representation is given in Eq. (2), where the vectors \( \delta x \), \( \delta u \), \( \delta w \) and \( \delta y \) are the dynamic deviations from the normal operating point. The input variables, \( u = [u_1, u_2, u_3, u_4]^T \), represent compressor speed, valve opening, condenser air flow rate, and evaporator air flow rate (see Fig. 1). The output variables are condenser pressure, condenser heat rejection rate, evaporator pressure, and evaporator cooling capacity, denoted as \( y = [P_c, Q_c, P_e, Q_e]^T \). The condenser and evaporator air inlet temperatures are regarded as disturbances: \( w = [T_{air,c}, T_{air,e}]^T \). The matrices \( A_i, B_i, C_i \) and \( F_i \) are obtained from the model linearization, and \( i = 1, 2, 3, 4 \) denotes the operation mode number. Note that one linear model is derived for each mode, and this paper focuses on the switching operation between different modes. However, multiple linearized models can also be developed for a single operation mode because the dynamics within each mode are themselves nonlinear.

\[
\delta x = A_i \delta x + B_i \delta u + F_i \delta w \\
\delta y = C_i \delta x 
\]

The dynamics of the linearized model are compared against those of the nonlinear model for each operation mode. One validation of the linearized model for mode 1 operation (see Fig. 2) is given here. Figure 4 shows the system input signals, and Figure 5 compares the dynamic response of condenser pressure, evaporator pressure, and evaporator air outlet temperature, of the linearized and nonlinear models. The dynamic responses match well, thereby validating the linearization.

![FIGURE 4. SYSTEM INPUT SIGNALS FOR LINEARIZED MODEL VALIDATION; DASHED LINE CORRESPONDS TO THE RIGHT Y-AXIS.](image)

### LMI-BASED CONTROL DESIGN

This section presents an LMI representation of LQR control design using the switched model framework as shown in Fig. 3.

In order to guarantee zero steady-state tracking error, the original plant model in Eq. (2) is augmented with integrated error states between the desired reference and the actual plant output as shown in Eq. (3) where \( y' \) is the output vector evaluated.
at a steady state operating point about which the system was linearized.

\[ z = \int (y - r) dt = \int (\dot{y} + \delta y - r) dt \]  (3)

The augmented state-space model representation is rewritten as

\[ \dot{\hat{x}} = \hat{A}_i \hat{x} + \hat{B}_i \delta u + \hat{F}_i \delta w + \hat{E}_i (y'_i - r'_i) \]  (4)

where

\[ \hat{X} = \begin{bmatrix} \delta x' \ \ z' \end{bmatrix}^T \]
\[ \hat{A}_i = \begin{bmatrix} A_i & 0 \\ C_i & 0 \end{bmatrix} \]
\[ \hat{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix} \]
\[ \hat{F}_i = \begin{bmatrix} F_i \\ 0 \end{bmatrix} \]
\[ \hat{E}_i = \begin{bmatrix} 0 \\ I \end{bmatrix} \]  (5)

The optimal LQR controller is obtained by finding the state-feedback gain \( K \), that minimizes the performance index

\[ J = \int_0^\infty (\hat{x}'^T Q \hat{x} + \delta u'^T R \delta u) dt. \]  (6)

Using LMI techniques [13], the LQR problem can be reformulated [14,15] as a convex optimization problem over \( P \) and \( M_i \)

\[ \min_{P,M_i,X} \text{Tr}(QP) + \text{Tr}(N) \]  (7)

subject to

\[ \begin{bmatrix} \hat{A}_i P + PA_i^T - \hat{B}_i M_i - M_i^T \hat{B}_i^T & M_i^T P^T \\ M_i & -R^{-1} & 0 \\ P & 0 & -Q^{-1} \end{bmatrix} < 0 \]  (8)

and

\[ \begin{bmatrix} N & R^{1/2} M_i \\ M_i^{T} R^{1/2} & P \end{bmatrix} > 0 \]  (9)

\[ P > 0. \]

Hence, the LMI-based LQR design problem can be summarized as the minimization of Eq. (7) under the constraints given by Eqs. (8) and (9). The optimal LQR controller is presented in Eq. (10). The main advantage of this formulation is that the solution provides a common Lyapunov matrix \( P^{-1} \) to ensure system stability during mode switching operations and achieve desired performance of the closed-loop system. The interested reader is referred to [14, 15] for a detailed derivation of the LMI-based LQR design problem.

\[ \delta u = -K \delta x \]  (10)

\[ K_i = M_i P^{-1} \]

Since some state variables in Eq. (1) cannot be measured directly (e.g., normalized two-phase zone length in the condenser and evaporator), a standard Kalman filter [16] is developed based on the linearized plant model in Eq. (2) for operation mode \( i \), as given below where \( \delta \hat{x} \) is an estimate of the state variable \( \delta x \).

\[ \begin{align*}
\delta \dot{\hat{x}} &= A_i \delta \hat{x} + B_i \delta u + F_i \delta w + L_i (\delta y_m - C_i \delta \hat{x}) \\
L_i &= S_i C_i^T V^{-1}
\end{align*} \]  (11)

The observer gain matrix \( L_i \) is computed from

\[ L_i = S_i C_i^T V^{-1} \]  (12)

where \( S_i \in \mathbb{R}^{13 \times 13} \) is the solution of the Riccati equation.
\[ S^T A_i + A_i S_i - S^T C_i V^{-1} C_i S_i + W = 0. \] (13)

The noise covariance matrices \( W \) and \( V \) in Eq. (13) are selected as the identity matrices, \( I^{3\times 3} \) and \( I^{2\times 2} \), respectively, in this paper. With the state observer, the MIMO feedback control diagram for each operation mode is presented in Fig. 6.

**CASE STUDY**

Building upon the control framework discussed above, a model-based switching control case study is presented here to demonstrate the effectiveness of the proposed approach. The model-based control scheme is shown in Fig. 7, where the local control network is composed of four individual controllers, and the switched model network is used to represent the dynamics of the nonlinear VCC system.

This section is divided into two parts. First, an optimization problem for determining optimal operation set-points is described. Second, for a particular VCC system, simulation results demonstrate the effectiveness of the mode switching control on achieving the optimal set-points.

**Set-point Optimization**

In order to determine the optimal operation set-points for the VCC system for a given performance requirement, a nonlinear optimization problem can be formulated and solved. One common efficiency objective is to minimize the total power consumption in the VCC system while meeting the desired cooling capacity. In this case, the objective function is

\[ J_{\text{power}} = \dot{W}_{\text{total}} = \dot{W}_{\text{compressor}} + \dot{W}_{\text{fan},e} + \dot{W}_{\text{fan},e} \] (14)

where

\[ \dot{W}_{\text{compressor}} = \frac{\dot{m}_{\text{ref}} (h_{\text{in},e} - h_{\text{out},e})}{\eta} \] (15)

\[ \eta = f_1(P_e, P_c, \omega) \] (16)

\[ \dot{W}_{\text{fan},e} = f_3(m_{\text{air},e}) \] (17)

and

\[ \dot{W}_{\text{fan},e} = f_3(m_{\text{air},e}). \] (18)

The functions \( f_1, f_2, \) and \( f_3 \) are empirical nonlinear functions that are specific to the particular VCC system that is being optimized. The vector of optimization variables is

\[ \nu = \begin{bmatrix} P_e & P_c & h_{\text{in},e} & \dot{m}_{\text{ref}} & m_{\text{air},e} \end{bmatrix} \] (19)

where

\[ \nu^* = \arg \min_{\nu} (J_{\text{power}}). \] (20)

A number of different constraints must be imposed in this optimization problem. The desired cooling capacity, \( C_{\text{des}} \), is enforced as a hard constraint on the system as shown in Eq. (21) where \( C_{\text{des}} \) is specified by the user.

\[ C_{\text{des}} = \dot{Q}_e = \dot{m}_{\text{ref}} (h_{\text{out},e} - h_{\text{in},e}) \] (21)

Additional inequality and equality constraints are required for the optimization, as listed in Eq. (22).

Note that \( T_{\text{ref},\text{sat},e} \) and \( T_{\text{ref},\text{sat},c} \) are the saturated refrigerant temperatures at \( P_e \) and \( P_c \), respectively. The first and second constraints in Eq. (22) ensure the correct temperature gradients at the outlet of the evaporator and condenser, respectively. The third and fourth constraints are modified for each individual operation mode (see Fig. 2). For example, in operation mode 3 or 4, the evaporator model is a one-zone (two-phase) representation. Therefore, the evaporator superheat, \( T_{\text{ref},\text{sat},c} - T_{\text{ref},\text{sat},e} \), is exactly zero. In operation mode 2 or 3, the condenser subcooled temperature, \( T_{\text{ref},\text{sat},c} - T_{\text{ref},\text{sat},e} \), becomes zero. This may result in a different optimal solution since the dynamics of the system are different when it is operating in this mode. As shown later in this paper, for a given cooling capacity, the optimization yields a ‘best’ operation mode with regards to the efficiency metric \( J_{\text{power}} \) defined in Eq. (14). The fifth constraint ensures that the first law of thermodynamics is not violated.
The final constraint listed in Eq. (22) ensures that the total amount of refrigerant mass is maintained in the system, independent of the mode in which the system is operating. $M_{\text{sys}}$ is the amount of mass estimated to be in the system based on a given set of values for the optimization variables, and $M_{\text{est}}$ is the total mass specified by the user. An error of 5% is allowed due to the effect of modeling uncertainties on the calculation of $M_{\text{est}}$. The dynamics of the VCC system are primarily driven by the refrigerant mass distribution throughout the system. Since the VCC system is closed, the total amount of mass remains constant at all times, and this physical constraint naturally limits how the system can operate.

The objective function is nonlinear; therefore, the function $fmincon$ in the MATLAB Optimization Toolbox is used to find a solution to the optimization problem.

**Controller Implementation**

The VCC system parameters in [2] are applied for the switched model network formulation and controller design. Via the set-point optimization described above (again applied to the system described in [2]), the best operation mode, with respect to minimal power consumption, can be found for a particular cooling capacity. A common efficiency metric for VCC systems is the coefficient of performance (COP) given by

$$\text{COP} = \frac{\dot{Q}_c}{\dot{W}_{\text{total}}} \quad (23)$$

where a higher COP indicates more efficient operation. Since $\dot{Q}_c$ is constrained to be equal to the desired cooling capacity $C_{\text{des}}$ (see Eq. (21)), minimizing $\dot{W}_{\text{total}}$ is equivalent to maximizing COP.

Figure 8 shows a comparison of the results in terms of COP among operation modes for different cooling capacity requirements: high (1.2 kW), medium (0.8 kW), and low (0.2 kW) capacity. The condenser and evaporator air inlet temperatures are set to be 26°C and 20°C, respectively, and the total amount of refrigerant mass in the system is specified as 135g. It can be seen that the minimum power consumption is achieved in operation mode 2 to meet the high cooling capacity, in mode 1 to meet the medium cooling capacity, and in mode 3 to meet the low cooling capacity. Table 1 summarizes the optimal set-points for the best operation mode at each capacity level, along with the desired system inputs and refrigerant mass distribution in each of the heat exchangers.

![Figure 8. Performance comparison between different operation modes for each of three desired cooling capacities.](image)

**TABLE 1. SUMMARY OF OPTIMIZATION RESULTS**

<table>
<thead>
<tr>
<th>Operation mode</th>
<th>High capacity</th>
<th>Medium capacity</th>
<th>Low capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cond. pressure (kPa)</td>
<td>1188</td>
<td>967.6</td>
<td>881.6</td>
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<tr>
<td>Evap. pressure (kPa)</td>
<td>315.5</td>
<td>290.4</td>
<td>472.3</td>
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<tr>
<td>Cond. subcooling (°C)</td>
<td>0</td>
<td>2.29</td>
<td>0</td>
</tr>
<tr>
<td>Evap. superheat (°C)</td>
<td>4.42</td>
<td>7.6</td>
<td>0</td>
</tr>
<tr>
<td>Comp. speed (rpm)</td>
<td>2000</td>
<td>1500</td>
<td>700</td>
</tr>
<tr>
<td>Valve opening (%)</td>
<td>19</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td>Cond. air flow (kg/s)</td>
<td>0.22</td>
<td>0.3</td>
<td>0.22</td>
</tr>
<tr>
<td>Evap. air flow (kg/s)</td>
<td>0.26</td>
<td>0.11</td>
<td>0.05</td>
</tr>
<tr>
<td>Mass in cond. (g)</td>
<td>116.8</td>
<td>114.6</td>
<td>94.7</td>
</tr>
<tr>
<td>Mass in evap. (g)</td>
<td>19.1</td>
<td>27.2</td>
<td>47.1</td>
</tr>
</tbody>
</table>

The LMI-based optimal control problem is solved to find the optimal state feedback gain matrix $K$ and observer gain matrix $L$ for each operation mode, $(K_1, L_1), (K_2, L_2)$, and $(K_3, L_3)$. In the LQR design, the weighting matrix $Q$ is chosen as the identity matrix $I_{17\times17}$, and the matrix $R$ is given in Eq. (24) to meet the system actuators’ saturation limits.

$$R = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10000 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10000 & 0 \\ 0 & 0 & 10000 & 0 & 0 & 0 & 0 & 0 & 0 & 10000 & 0 \end{bmatrix} \quad (24)$$
The desired cooling capacity profile is given in Fig. 9, and Fig. 10 shows the corresponding operation mode switching sequence. The controller performance while tracking the optimal set-points (condenser pressure, condenser heat rejection rate, evaporator pressure, and evaporator cooling capacity) is presented in Fig. 11. It is shown that the designed controllers can maintain system operation at the optimal set-points with mode switching.

Figures 12 and 13 show the control input signals to the VCC system. It is evident by the large changes in compressor speed, valve opening, and heat exchanger air flow rates that the system is being effectively controlled over a very large operating region. As illustrated, the proposed switching control scheme drives the system from one operation mode to another based on the determined optimal set-points listed in Tab. 1. It should be noted that control techniques, such as bumpless transfer [17] or dynamic compensator design [18], can be applied to ensure smoother control signals during mode switching.

CONCLUSIONS

This paper presents a switched model network to represent the nonlinear VCC system dynamics over the normal operating envelope. Four different operation modes are introduced to formulate a 13th-order switched model of the system. The proposed LMI control design framework guarantees the stability and performance characteristics of the controlled closed-loop system. A set-point optimization problem is solved to determine the optimal set-points for achieving a desired cooling capacity profile for a particular VCC system. A model-based switching
controller is successfully designed to regulate the system performance across the nonlinear operating envelope.

Future work will involve comparison studies between the proposed model and control framework and conventional VCC system control approaches as well as finding optimal sequences for mode switching in the presence of disturbances and other exogenous signals.

ACKNOWLEDGMENTS
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REFERENCES