Fundamentals of Engineering Exam

2009 Review Session

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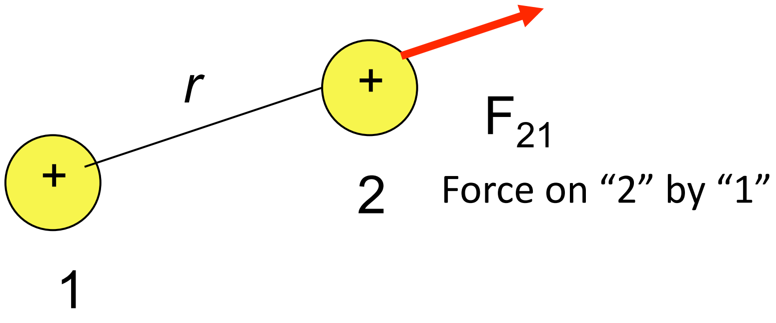
Electricity & Magnetism

- Coulomb's Law
- Electrostatic Potential Energy
- Electrostatic Potential – Voltage
- Magnetic Force
- Electric Current
- Current & Voltage laws
- Resistive Circuits
- Capacitance & Inductance
- AC Circuits

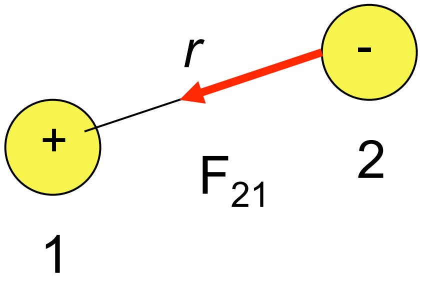
The Coulomb Force Law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2}\hat{r}$$

ε_0 = permittivity constant



Force repulsive



Force attractive

- **The force exerted by one point charge on another acts along line joining the charges.**
- **The force is repulsive if the charges have the same sign and attractive if the charges have opposite signs.**

Units and Constants

SI units of electric charge: Coulomb, C

Constants:

$1/4\pi\varepsilon_0 = 9 \times 10^9$ N·m²/C²

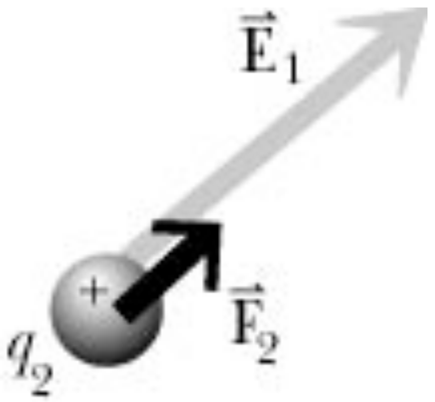
$\varepsilon_0 = 8.85 \times 10^{-12}$ C²/N·m² permittivity constant

$e = 1.602 \times 10^{-19}$ C

1 C = 6.24×10^{18} elementary charges

$$\left|\vec{F}\right| = F = \frac{1}{4\pi\varepsilon_0} \frac{\left|Q_1 Q_2\right|}{r^2}$$

Definition of Electric Field



$$F = \frac{1}{4\pi\varepsilon_0}\frac{|q_1 q_2|}{r^2}$$

$$F = q_2\left(\frac{1}{4\pi\varepsilon_0}\frac{|q_1|}{r^2}\right)$$

$$\vec{F}_2 = q_2\vec{E}_1$$

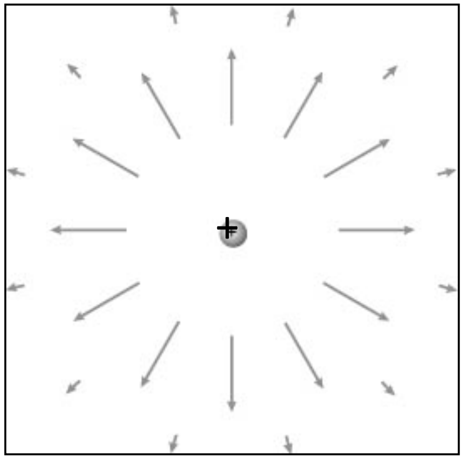
$$\vec{E}_1 = \vec{F}_2 / q_2$$

The Electric Field of a Point Charge

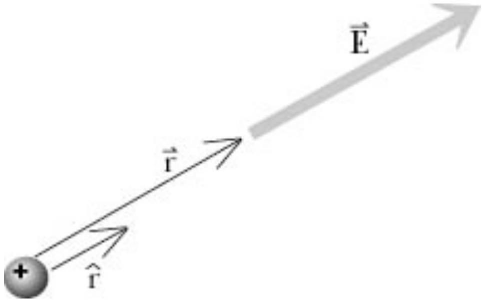
$$F = q_2\left(\frac{1}{4\pi\varepsilon_0}\frac{|q_1|}{r^2}\right)$$

$$\vec{F}_2 = q_2\vec{E}_1$$

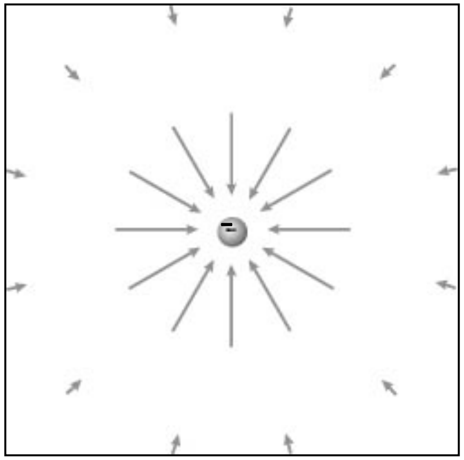
$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{|q_1|}{r^2}$$



Including direction:



$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}\hat{r}$$



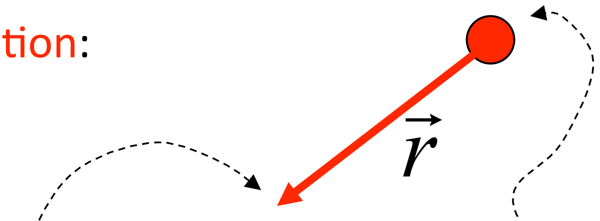
Example Problem

A particle with charge +2 nC (1 nanoCoulomb=10^{-9} C) is located at the origin. What is the electric field due to this particle at a location <-0.2,-0.2,-0.2> m?

Solution:

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}\hat{r}$$

1. Distance and direction:



$$\vec{r} = \langle observed\_location \rangle - \langle source\_location \rangle$$

$$\vec{r} = \langle -0.2,-0.2,-0.2 \rangle - \langle 0,0,0 \rangle = \langle -0.2,-0.2,-0.2 \rangle$$

$$|\vec{r}| = \sqrt{(-0.2)^2 + (-0.2)^2 + (-0.2)^2} = 0.35 \text{ m}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle -0.2,-0.2,-0.2 \rangle}{0.35} = \langle -0.57,-0.57,-0.57 \rangle$$

Example Problem

2. The magnitude of the electric field:

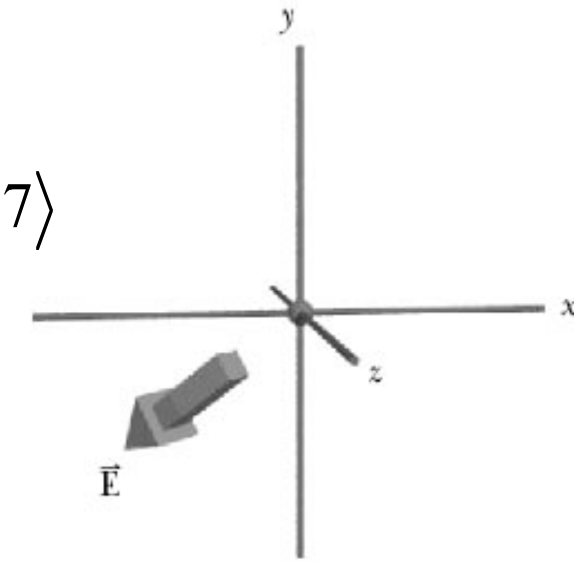
$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}\hat{r}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \left(9\times10^9\,\frac{\text{Nm}^2}{\text{C}^2}\right)\left(\frac{2\times10^{-9}\,\text{C}}{0.35^2\,\text{m}^2}\right) = 147\,\frac{\text{N}}{\text{C}}$$

3. The electric field in vector form:

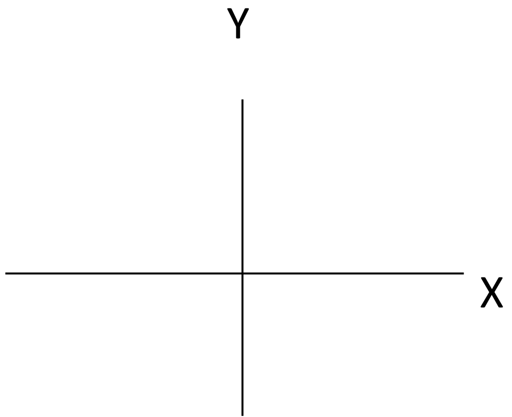
$$\vec{E} = E\hat{r} = \left(147\,\frac{\text{N}}{\text{C}}\right)\langle -0.57, -0.57, -0.57\rangle$$

$$\vec{E} = \langle -84, -84, -84\rangle\frac{\text{N}}{\text{C}}$$



Forces due to an Electric Field

Example: The electric field at a particular location is <-300,0,0> N/C. What force would an electron experience if it were placed in this location?



Solution:

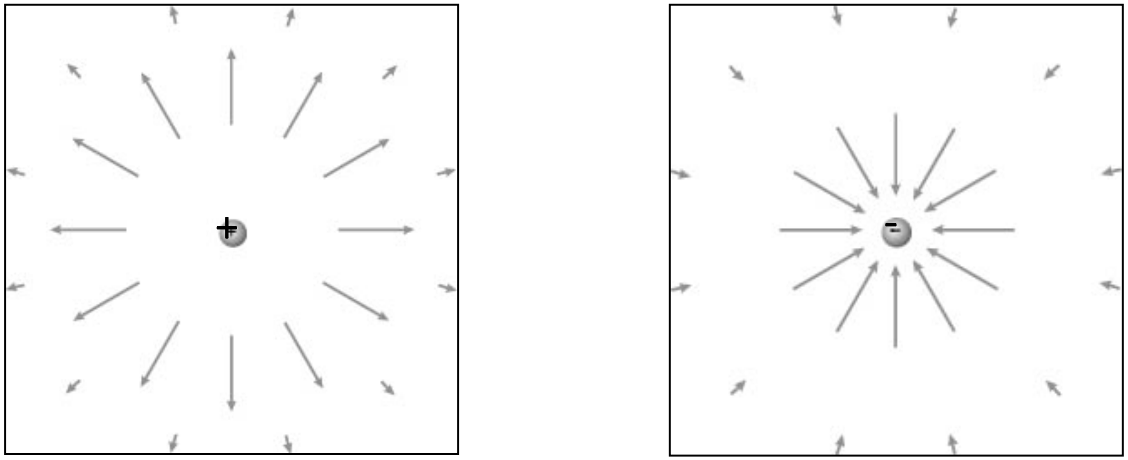
$$\vec{F} = -e\vec{E} = -1.6\times10^{-19}\,\text{C}\langle -300,0,0\rangle\,\text{N/C}$$

$$\vec{F} = \langle 4.8\times10^{-17},0,0\rangle\,\text{N}$$

The Electric Field

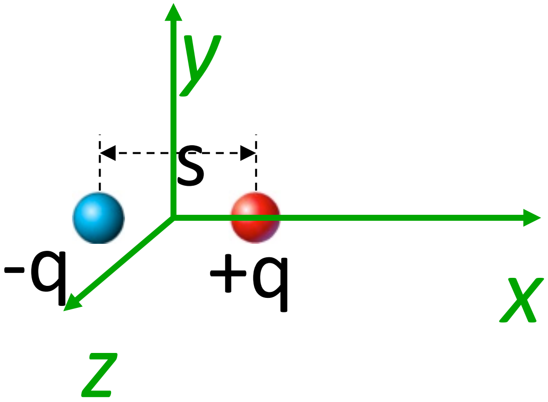
Point Charge:

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}\hat{r}$$



Dipole:

for r>>s :



$$\vec{E} = \left\langle \frac{1}{4\pi\varepsilon_0}\frac{2qs}{r^3}, 0, 0 \right\rangle$$ at <r,0,0>

$$\vec{E} = \left\langle \frac{-1}{4\pi\varepsilon_0}\frac{qs}{r^3}, 0, 0 \right\rangle$$ at <0,r,0>

$$\vec{E} = \left\langle \frac{-1}{4\pi\varepsilon_0}\frac{qs}{r^3}, 0, 0 \right\rangle$$ at <0,0,r>

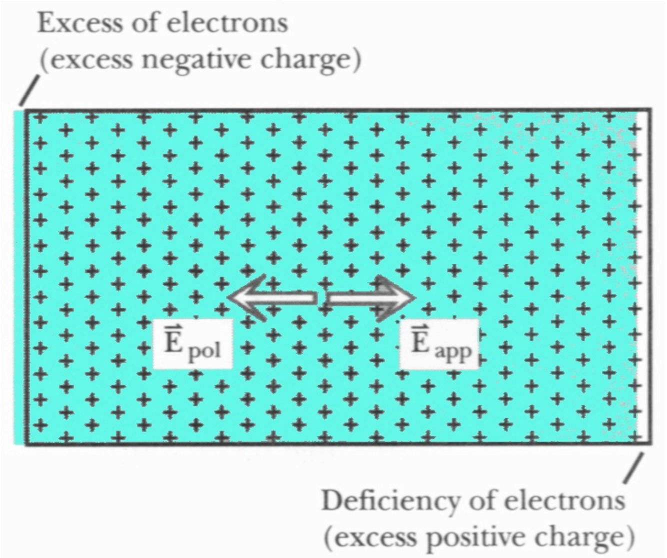
Conductors and Insulators

Different materials respond differently to electric field

Conductor: contains mobile charges that can move through material

Insulator: contains no mobile charges

Electric Field Inside Metal



In static equilibrium:

$$\vec{E}_{net} = \vec{E}_{app} + \vec{E}_{pol} = 0$$

E_{net}= 0 everywhere inside the metal!

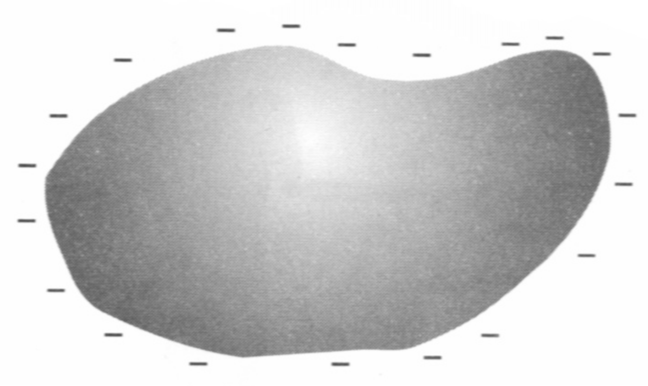
Mobile charges on surface rearrange to achieve E_{net}= 0

Actual arrangement might be very complex!

It is a consequence of $1/r^2$ distance dependence

E_{net}= 0 only in static equilibrium!

Excess Charge on Conductors



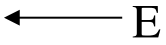
Excess charges in any conductor are always found on an inner or outer surface!

Conductors versus Insulators

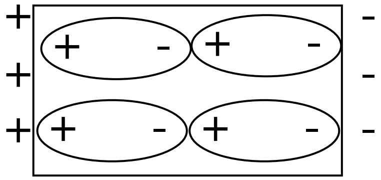
	Conductor	**Insulator**
Mobile charges	yes	no
Polarization	entire sea of mobile charges moves	individual atoms/molecules polarize
Static equilibrium	E_{net}= 0 inside	E_{net} nonzero inside
Excess charges	only on surface	anywhere on or inside material
Distribution of excess charges	Spread over entire surface	located in patches

Question

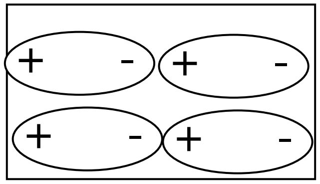
An electric field polarizes a metal block as shown below. Select the diagram that represents the final state of the metal.



A)

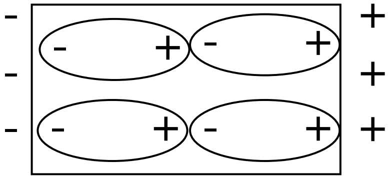


C)



B)

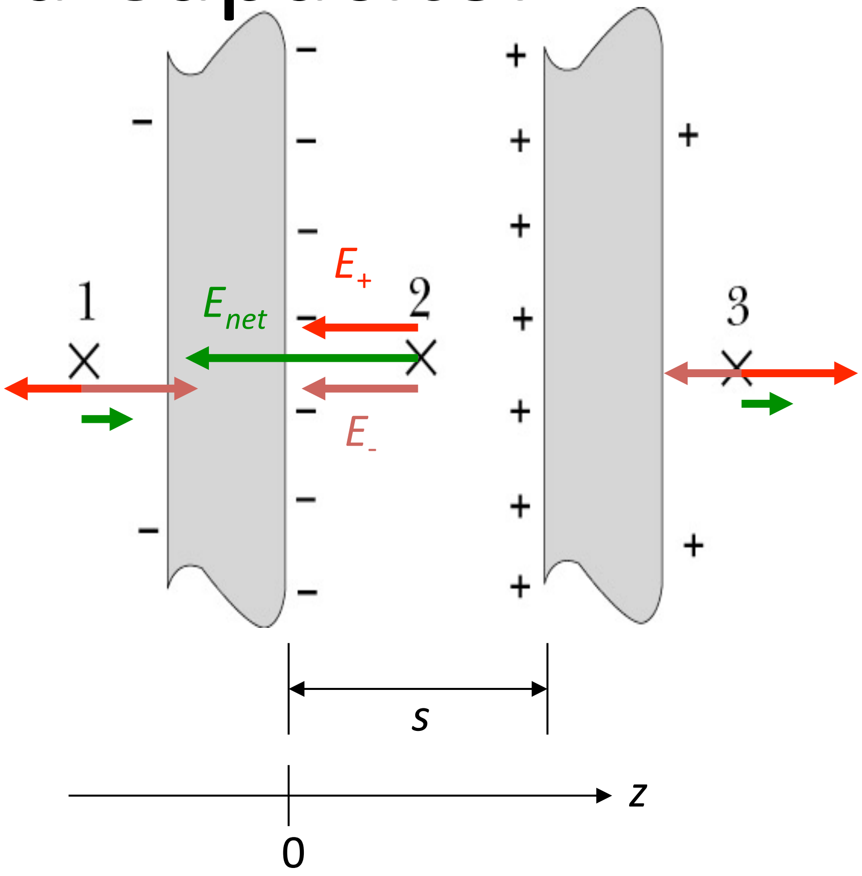
D)



Electric Field of a Capacitor

Inside: $$E_2 \approx \frac{Q/A}{\varepsilon_0}$$

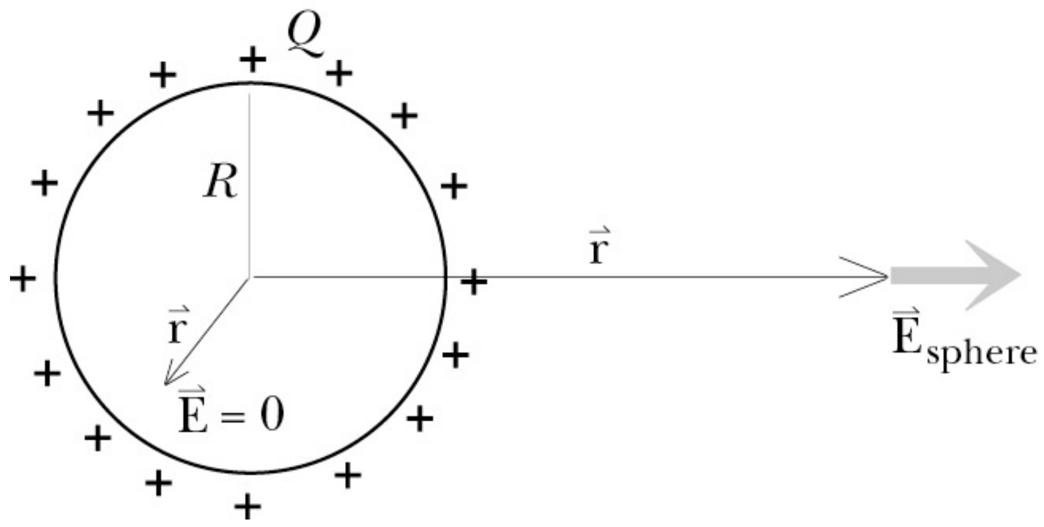
Fringe: $$E_1 = E_3 \approx \frac{Q/A}{2\varepsilon_0}\frac{s}{R}$$



Step 4: check the results:

☑ Units: $$\frac{\mathrm{C/m^2}}{\mathrm{C^2/(N\cdot m^2)}} = \frac{\mathrm{N}}{\mathrm{C}}$$

Electric Field of a Spherical Shell of Charge

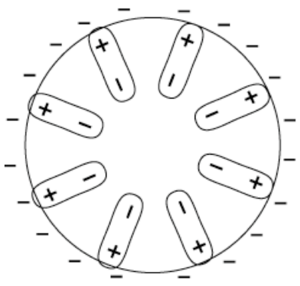


Field inside: $\vec{E} = 0$

Field outside: $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{r}$ (like point charge)

A solid plastic ball has been rubbed all over with a piece of wool so that negative charge is uniformly spread over its surface. Which diagram best shows the polarization of molecules inside the ball?

1



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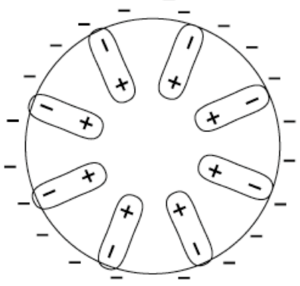

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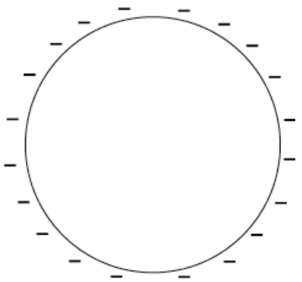
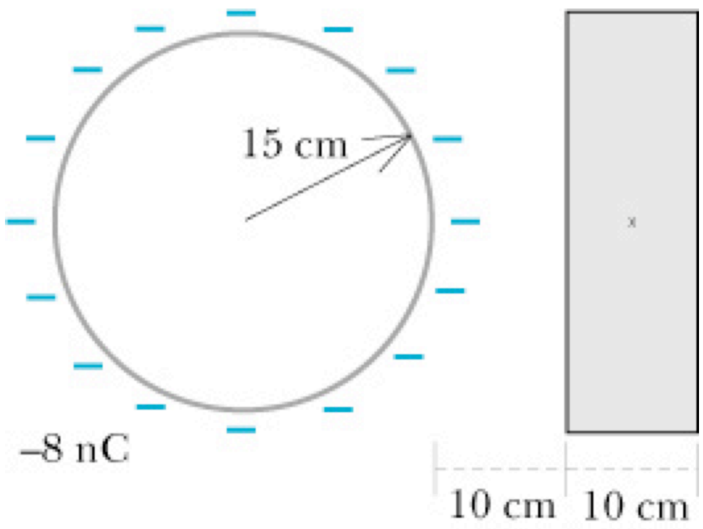


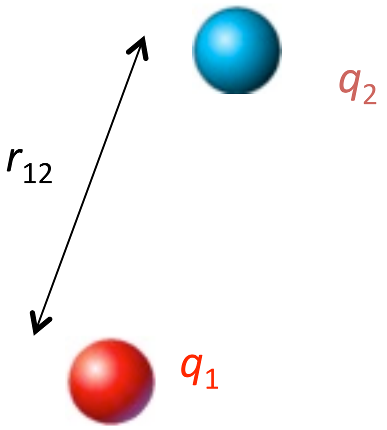


A very thin spherical plastic shell of radius 15 cm carries a uniformly distributed negative charge of -8 nC on its outer surface. An uncharged solid metal block is placed nearby. The block is 10 cm thick, and it is 10 cm away from the surface of the sphere.
The magnitude of the electric field at the center of the metal block due only to the charges on the block itself is:

1. 1152 N/C
2. 3200 N/C
3. 0 N/C
4. 800 N/C
5. 1800 N/C

Electric Potential Energy of Two Particles

Potential energy is associated with pairs of interacting objects



Energy of the system:

1. Energy of particle q_1 = E_1
2. Energy of particle q_2 = E_2
3. Interaction energy U_{el}

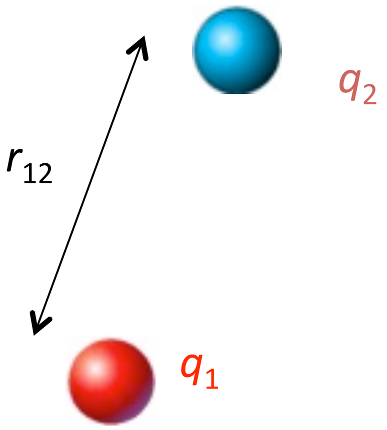
$$E_{system} = E_1+E_2+U_{el}$$

To change the energy of particles we have to perform work.

$$\Delta E_1 + \Delta E_2 = W_{ext} + W_{\text{int}} + Q$$

W_{ext} – work done by forces exerted by other objects
W_{int} – work done by electric forces between q_1 and q_2
Q – thermal transfer of energy into the system

Electric Potential Energy of Two Particles



$$\Delta E_1 + \Delta E_2 = W_{ext} + W_{\text{int}} + Q$$

$$\Delta E_1 + \Delta E_2 - W_{\text{int}} = W_{ext} + Q$$

$$\Delta U_{el} \equiv -W_{int}$$

if $\Delta(mc^2) = 0$

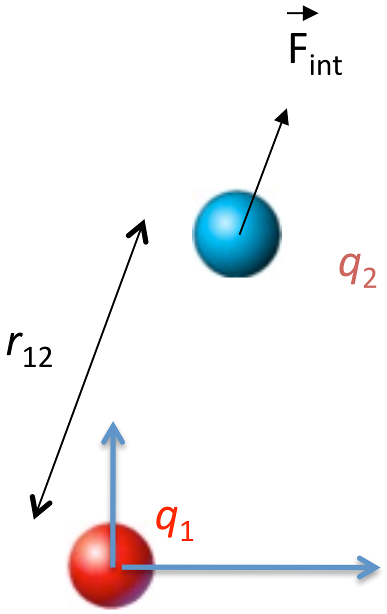
$$\Delta K_{system} + \Delta U_{el} = W_{ext} + Q$$

Total energy of the system can be changed (*only*) by external forces.

Work done by internal forces:

$$\Delta U_{el} = -W_{\text{int}} = -\int_i^f \vec{F}_{\text{int}} \bullet d\vec{r}$$

Electric Potential Energy of Two Particles



$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12}$$

$$\Delta U_{el} = -W_{int} = -\int_i^f \vec{F}_{int} \bullet d\vec{r}$$

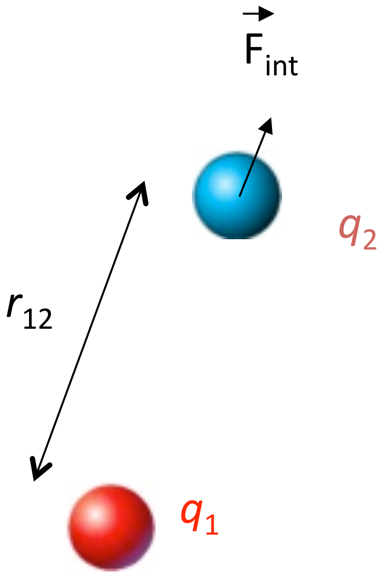
$$\Delta U_{el} = -\int_i^f \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} \bullet d\vec{r}_{12}$$

$$\Delta U_{el} = -\int_i^f \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2} dr_{12}$$

$$\Delta U_{el} = -\frac{1}{4\pi\varepsilon_0} q_1 q_2 \int_i^f \frac{1}{r_{12}^2} dr_{12}$$

$$\Delta U_{el} = -\frac{1}{4\pi\varepsilon_0} q_1 q_2 \left[-\frac{1}{r_{12}}\right]_i^f$$

Electric Potential Energy of Two Particles



$$\Delta U_{el} = -\frac{1}{4\pi\varepsilon_0} q_1 q_2 \left[-\frac{1}{r_{12}} \right]_i^f$$

$$\Delta U_{el} = \Delta\left(\frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}} \right)$$

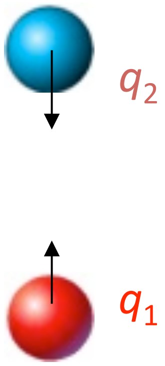
$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12}$$

The potential energy of a pair of particles is:

$$U_{el} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}} \text{ (joules)}$$

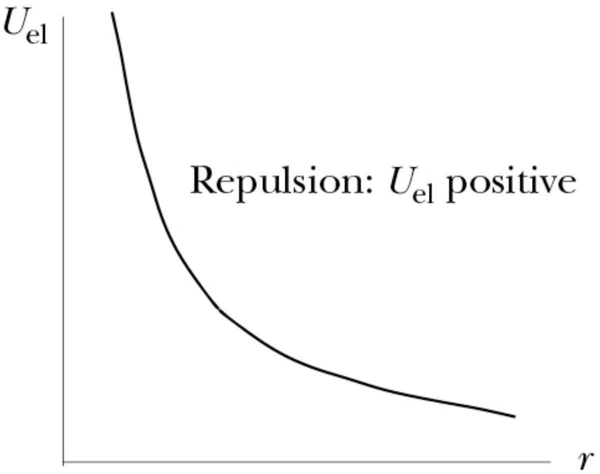
Electric Potential Energy of Two Particles

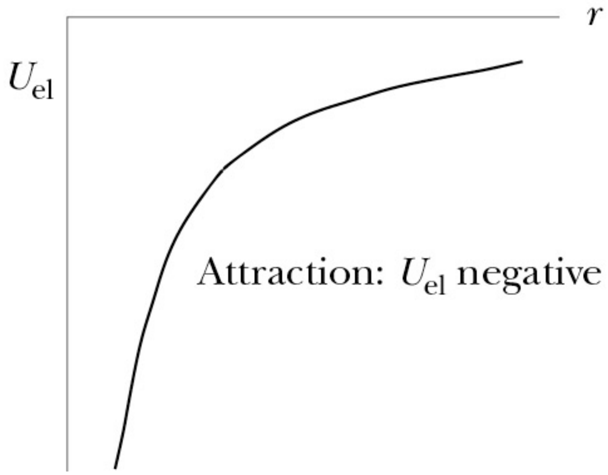
$$U_{el} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}} \text{ (joules)}$$



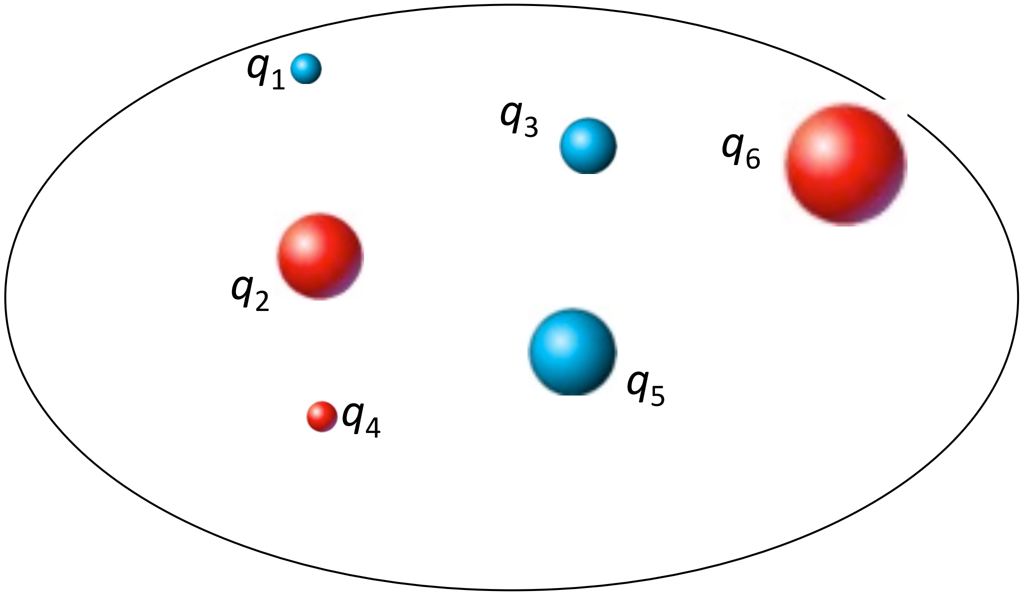
$U_{el} > 0$ for two like-sign charges (repulsion)

$U_{el} < 0$ for two unlike-sign Charges (attraction)





Multiple Electric Charges



Each (i,j) pair interacts: potential energy U_{ij}

$$U_{el} = \sum_{i<j} U_{ij} = \sum_{i<j} \frac{1}{4\pi\varepsilon_0} \frac{q_i q_j}{r_{ij}}$$

Electric Potential

Electric potential ≡ electric potential energy per unit charge

$$\boxed{V = \frac{U_{el}}{q}}$$

Alessandro Volta (1745 - 1827)

Units: J/C = V (Volt)

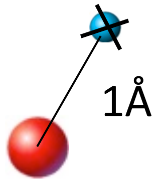
Volts per meter = Newtons per Coulomb

Electric potential – often called potential

Electric potential difference – often called voltage

Exercise

What is the electrical potential at a location 1Å from a proton?

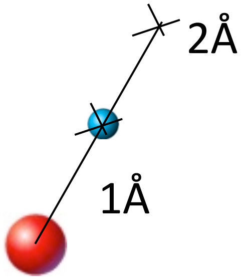


$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r} = \left(9\times10^9\,\frac{\mathrm{Nm}^2}{\mathrm{C}^2}\right)\frac{\left(1.6\times10^{-19}\,\mathrm{C}\right)}{\left(10^{-10}\,\mathrm{m}\right)} = 14.4\ \mathrm{J/C} = 14.4\ \mathrm{V}$$

What is the potential energy of an electron at a location
1Å from a proton?

$$U_{el} = Vq = (14.4\ \mathrm{J/C})\left(-1.6\times10^{-19}\ \mathrm{C}\right) = -2.3\times10^{-18}\ \mathrm{J}$$

Exercise



What is the change in potential in going from 1Å to 2Å from the proton?

$$\Delta V = V\left(2\,\mathring{\text{A}}\right) - V\left(1\,\mathring{\text{A}}\right) = -7.2\ \text{V}$$

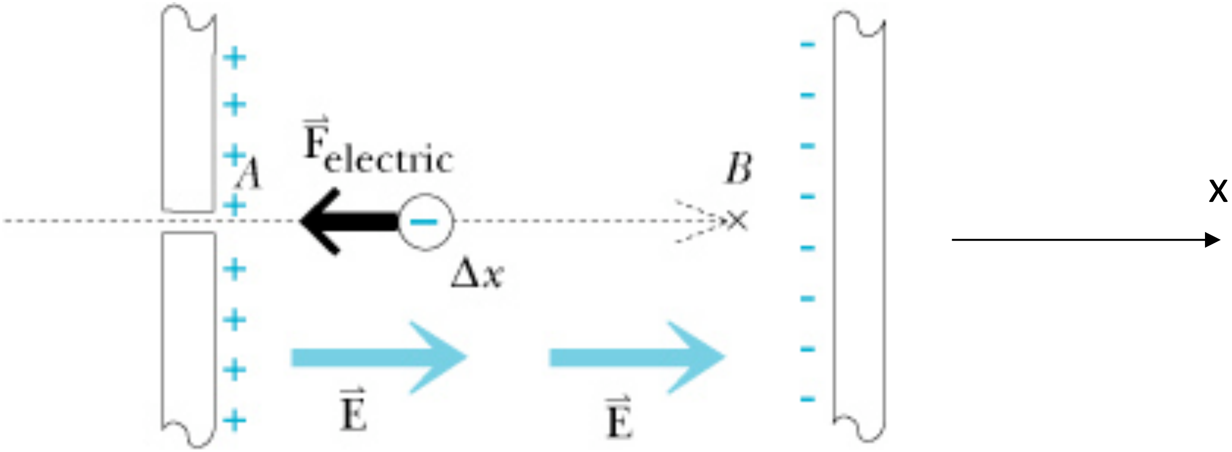
What is the change in electric potential energy associated with moving an electron from 1Å to 2Å from the proton?

$$\Delta U_{el} = U_{el}\left(2\,\mathring{\text{A}}\right) - U_{el}\left(1\,\mathring{\text{A}}\right) = qV\left(2\,\mathring{\text{A}}\right) - qV\left(1\,\mathring{\text{A}}\right) = q\Delta V$$

$$\Delta U_{el} = \left(-1.6\times 10^{-19}\ \text{C}\right)\left(-7.2\ \text{J/C}\right) = +1.15\times 10^{-18}\ \text{J}$$

Does the sign make sense?

Example



An electron traveling to the right enters capacitor through a small hole at A. Electric field strength is $2\text{x}10^3$ N/C. What is the change in the electron's potential energy in traveling from A to B? What is its change in kinetic energy? Δ(AB)= 4mm

$$\Delta U_{electric} = -\vec{F}_{\text{int}} \bullet \Delta\vec{l} = -(-eE_x)\Delta x = eE_x\Delta x$$

$$= (1.6\text{x}10^{-19}\ \text{C})(2\text{x}10^{3}\ \text{N/C})(0.004\text{m}) = 1.3\text{x}10^{-18}\ \text{J}$$

$$\Delta \text{K} = -\Delta \text{U}_{electric} = -1.3\text{x}10^{-18}\ \text{J}$$

Sign of the Potential Difference

$$\Delta U_{el} = q\Delta V$$

The potential difference ΔV can be positive or negative.
The sign determines whether a particular charged particle will gain or lose energy in moving from one place to another.

If $q\Delta V < 0$ – then potential energy decreases and K increases

If $q\Delta V > 0$ – then potential energy increases and K decreases

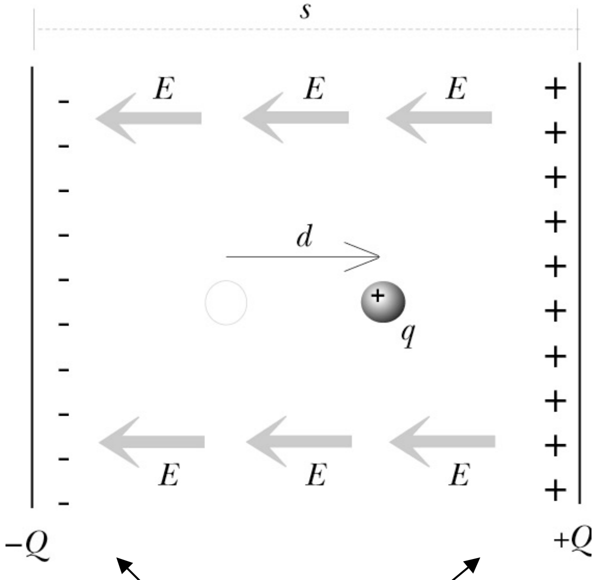
Path going in the direction of $\vec{E}$: Potential is decreasing ($\Delta V < 0$)

Path going opposite to $\vec{E}$: Potential is increasing ($\Delta V > 0$)

Path going perpendicular to $\vec{\mathrm{E}}$: Potential does not change ($\Delta V = 0$)

Sign of the Potential Difference

$$\Delta U_{el} = q\Delta V$$



$V_1 < V_2$

To move a positive charge to the area with higher potential:

$$V_f - V_i > 0$$

$$\Delta U_{el} = q\Delta V > 0$$

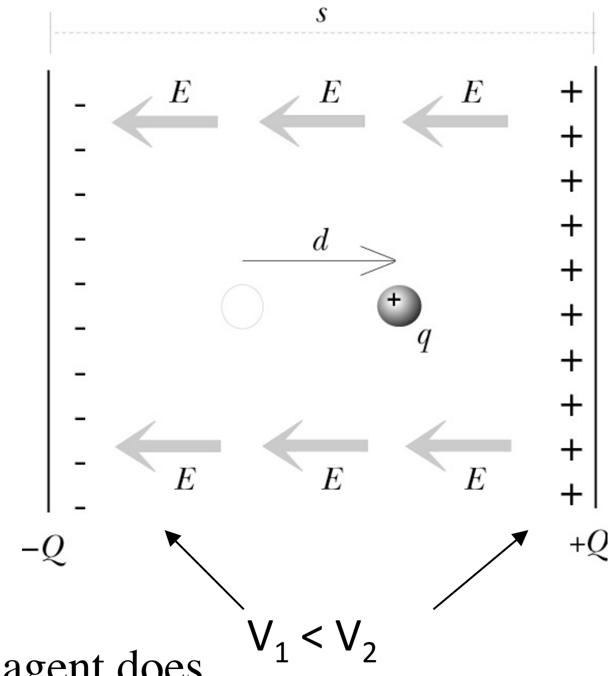
Need external force to perform work

Moving opposite to $\vec{E}$ means that potential is increasing

Question

A system consists of a proton inside of a capacitor. The proton moves from left to right as shown at a constant speed due to the action of an external agent.

Which of the following statements is true?



A) The proton's potential energy is unchanged and the external agent does no work on the system.

B) The proton's potential energy decreases and the external agent does work W > 0 on the system.

C) The proton's potential energy decreases and the external agent does work W < 0 on the system.

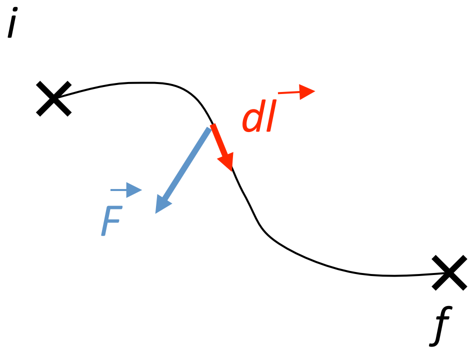
D) The proton's potential energy increases and the external agent does work W < 0 on the system.

E) The proton's potential energy increases and the external agent does work W > 0 on the system.

Potential Difference with Varying Field

$$\Delta V = -\int_{i}^{f} \vec{E} \bullet d\vec{l} = -\int_{x_i}^{x_f} E_x dx - \int_{y_i}^{y_f} E_y dy - \int_{z_i}^{z_f} E_z dz$$

Potential Difference and Electric Field



$$\Delta U = -W_{\text{int}} = -\int_i^f \vec{F}_{\text{int}} \bullet d\vec{l}$$

$$\Delta\left(\frac{U}{q}\right) = -\int_i^f \left(\frac{\vec{F}_{\text{int}}}{q}\right) \bullet d\vec{l}$$

$$\Delta V = -\int_i^f \vec{E} \bullet d\vec{l}$$

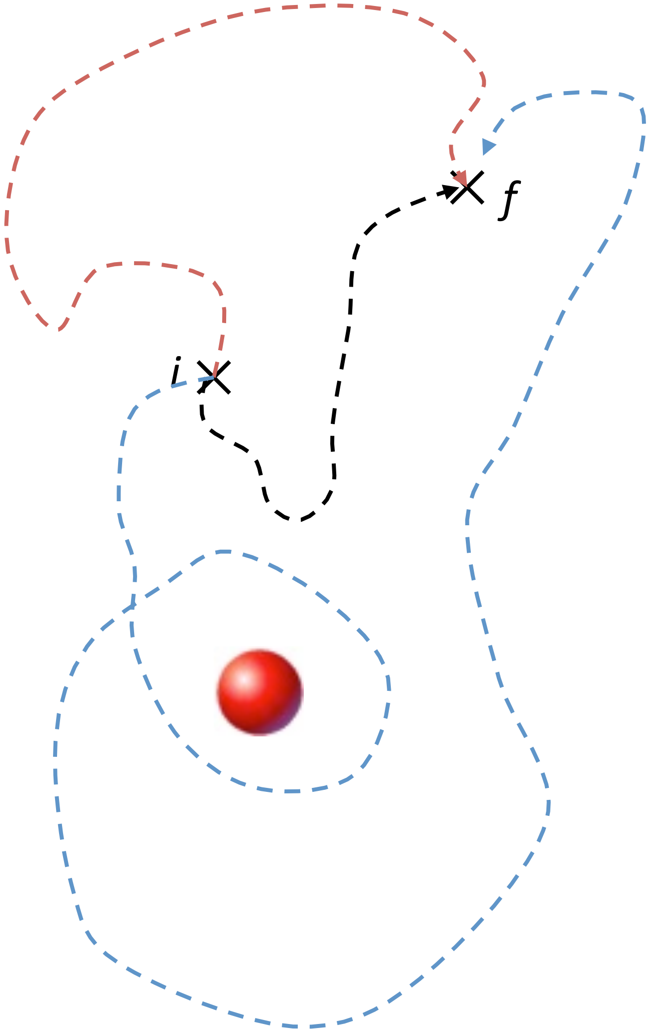
$$V_f - V_i = -\int_i^f \vec{E} \bullet d\vec{l}$$

For very short path: $\Delta V = -\vec{E} \bullet \Delta\vec{l}$

Example: E = 3·10^6 N/C, Δl = 1 mm:

$$\Delta V = -\left(3\times10^6 \text{ N/C}\right)\left(0.001 \text{ m}\right) = -3000 \text{ V}$$

Potential Difference: Path Independence



$$\Delta V = -\int_i^f \vec{E} \bullet d\vec{l}$$

$$\Delta V = V_f - V_i \qquad V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

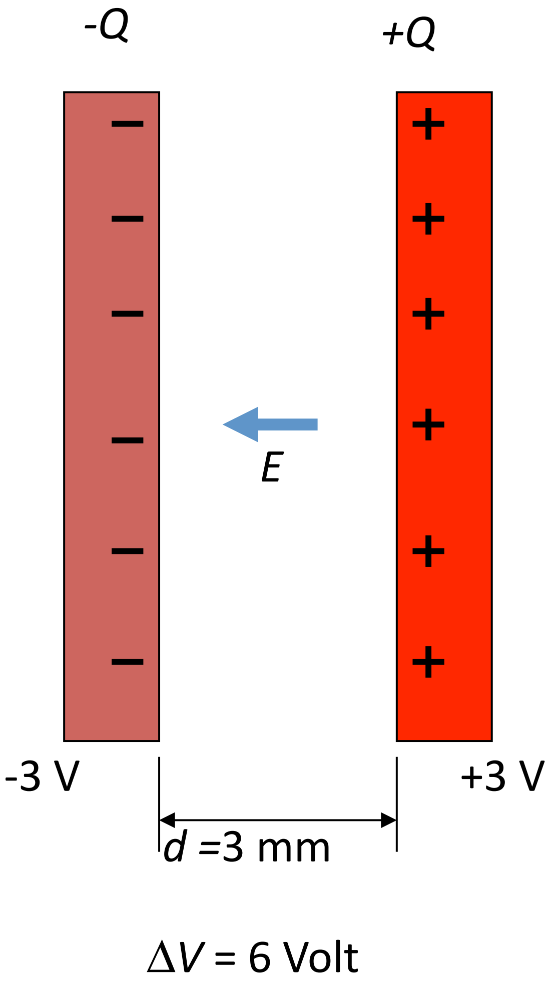
Path independence principle:

ΔV between two points does not depend on integration path

Potential in Metal

In static equilibrium

A Capacitor with large plates and a small gap of 3 mm has a potential difference of 6 Volts from one plate to the other.



$$E \approx \frac{Q/A}{\varepsilon_0}$$

$$\Delta V = -\int_i^f \vec{E} \bullet d\vec{l}$$

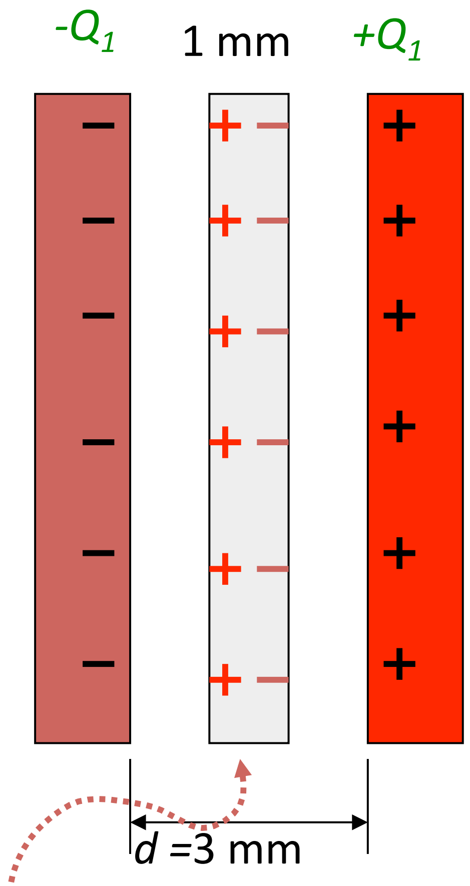
$$\Delta V = Ed = 6\ \text{V}$$

$$E = (6\ \text{Volts})/(0.003\ \text{m}) = 2000\ \text{Volts/m}$$

Charges are on surface

Potential in Metal

In static equilibrium



ΔV = 4 V

Charges $+Q_2$ and $-Q_2$

Insert a 1 mm thick metal slab into the center of the capacitor.

Metal slab polarizes and has charges $+Q_2$ and $-Q_2$ on its surfaces.

What are the charges Q_1 and Q_2?

$$E_1 \approx \frac{Q_1 / A}{\varepsilon_0} \qquad E_2 \approx \frac{Q_2 / A}{\varepsilon_0}$$

E inside metal is zero → $Q_2 = Q_1$

Now we have 2 capacitors instead of one

$$\Delta V = -\int_i^f \vec{E} \bullet d\vec{l}$$

$$\Delta V_{left} = \Delta V_{right} = \left(2000\ \text{V/m}\right)\left(0.001\ \text{m}\right) = 2\ \text{V}$$

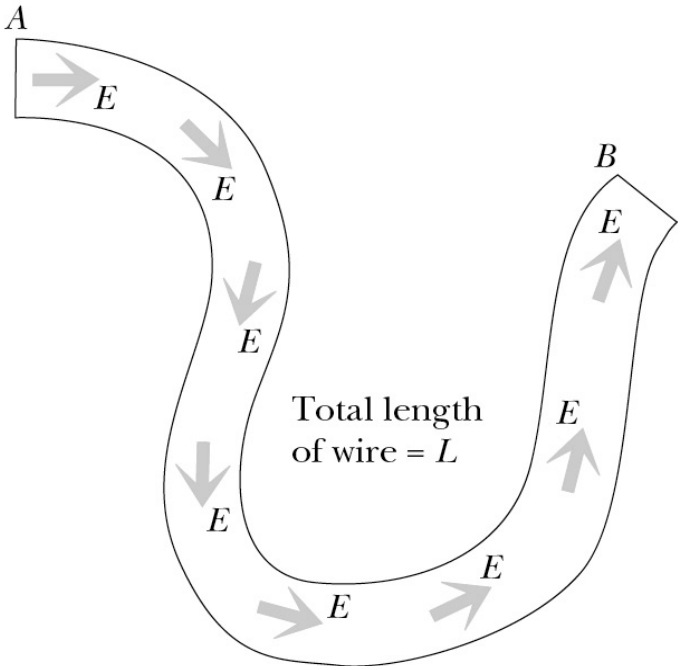
ΔV inside metal slab is zero!

Potential in Metal

Not in static equilibrium

Metal is not in static equilibrium:

- When it is in the process of being polarized
- When there is an external source of mobile charges (battery)



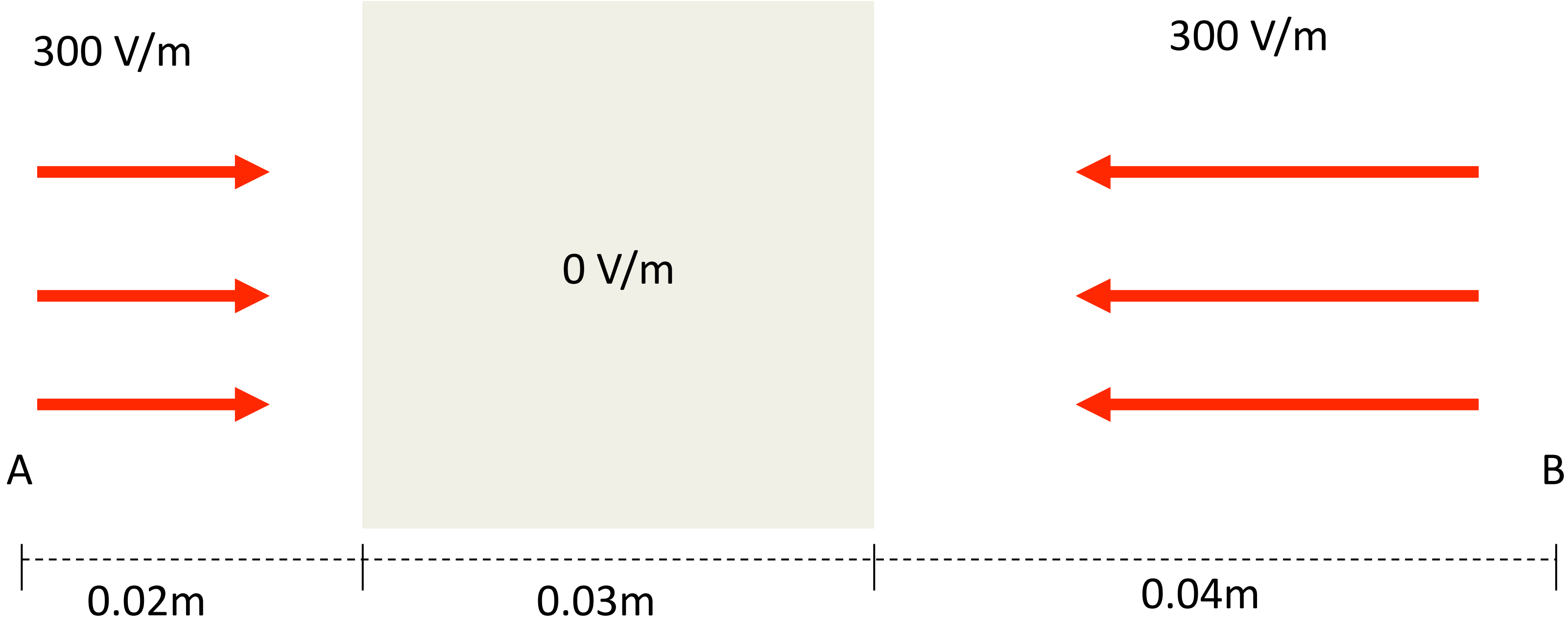
$$\Delta V = -\int_i^f \vec{E} \bullet d\vec{l}$$

For each step $\vec{E} \,||\, \Delta\vec{l}$, the potential difference is: $\Delta V = -EL$

If a metal is not in static equilibrium, the potential isn't constant in the metal.

Nonzero electric field of uniform magnitude *E* throughout the interior of a wire of length *L*.
Direction of the field follows the direction of the wire.

Question



What is V_B-$V_{A?}$

A) 270 V
B) -270 V
C) -18 V
D) 6 V
E) -6 V

Capacitance

Electric field in a capacitor:

$$E = \frac{Q/A}{\varepsilon_0}$$

$$\Delta V = -\int_i^f \vec{E} \bullet d\vec{l} \longrightarrow |\Delta V| = Es$$

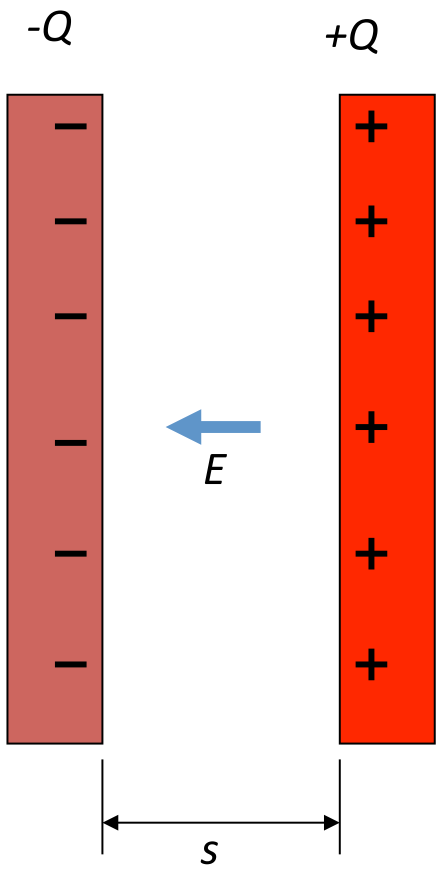
$$|\Delta V| = \frac{Q/A}{\varepsilon_0} s \longrightarrow Q = \frac{\varepsilon_0 A}{s} |\Delta V|$$

In general: $Q \sim |\Delta V|$

Definition of capacitance:

$$Q = \underset{\uparrow}{C} |\Delta V|$$

Capacitance



Capacitance of a parallel-plate capacitor:

$$\boxed{C = \frac{\varepsilon_0 A}{s}}$$

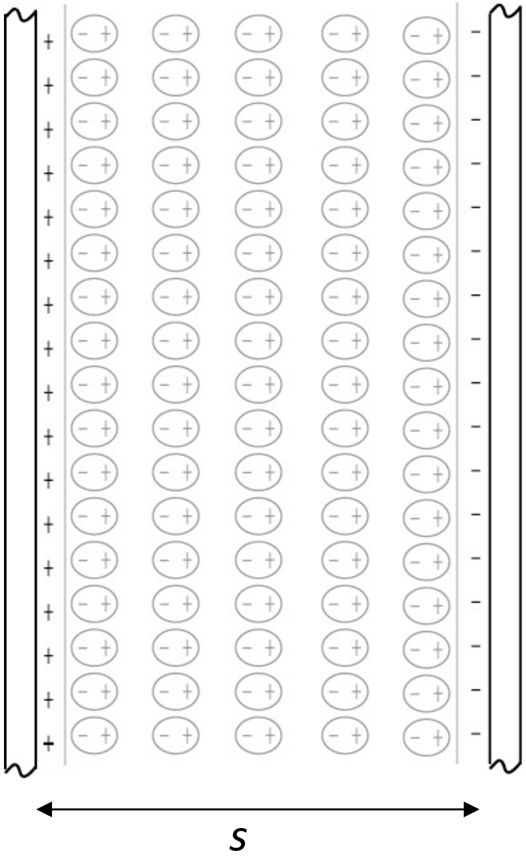
Capacitance

$$Q = C\left|\Delta V\right|$$

Units: C/V, Farads (F)

Michael Faraday
(1791 - 1867)

Potential Difference in a Capacitor with Insulator



$$\Delta V = -\int_A^B \vec{E} \bullet d\vec{l}$$

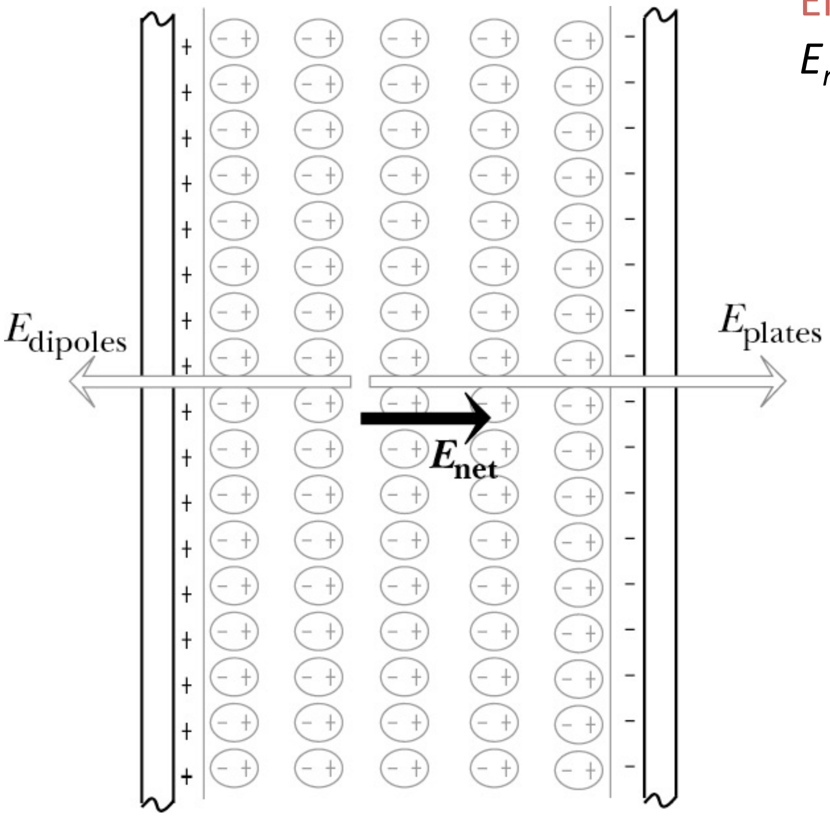
$$E_{plates} = \frac{(Q/A)}{\varepsilon_0}$$

$$|\Delta V| = Es = \frac{E_{plates}}{K} s$$

$$|\Delta V| = Es = \frac{(Q/A)}{K\varepsilon_0} s$$

$$\Delta V_{insulator} = \frac{\Delta V_{vacuum}}{K}$$

Dielectric Constant



Electric field in capacitor filled with insulator:

$E_{net}=E_{plates}-E_{dipoles}$

K – dielectric constant

$$E_{net}=\frac{E_{plates}}{K}$$

$$E_{plates}=\frac{(Q/A)}{\varepsilon_0}$$

$$E_{net}=\frac{(Q/A)}{K\varepsilon_0}$$

A Capacitor With an Insulator Between the Plates

No insulator:

$$\boxed{E = \frac{Q/A}{\varepsilon_0}}$$

$$|\Delta V| = Es$$

$$|\Delta V| = \frac{Q/A}{\varepsilon_0} s$$

$$Q = \frac{\varepsilon_0 A}{s} |\Delta V|$$

$$\downarrow$$

$$C = \frac{\varepsilon_0 A}{s}$$

With insulator:

$$\boxed{E = \frac{Q/A}{K\varepsilon_0}}$$

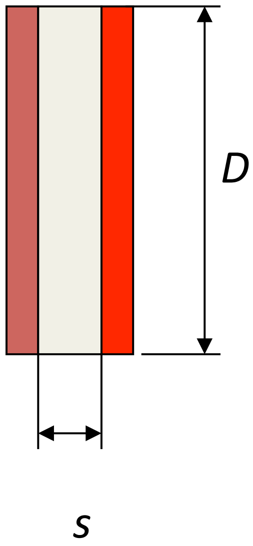
$$|\Delta V| = Es$$

$$|\Delta V| = \frac{Q/A}{K\varepsilon_0} s$$

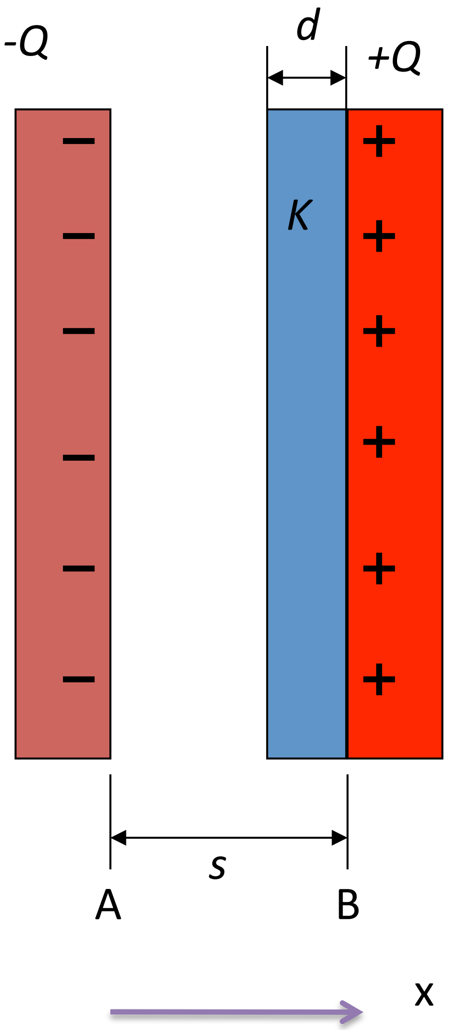
$$Q = \frac{K\varepsilon_0 A}{s} |\Delta V|$$

$$\downarrow$$

$$C = K \frac{\varepsilon_0 A}{s}$$



Potential Difference in Partially Filled Capacitor



$$\Delta V = -\int_A^B \vec{E} \bullet d\vec{l}$$

$$\vec{E}_{plates} = -\frac{(Q/A)}{\varepsilon_0}\hat{x}$$

$$\Delta V = \Delta V_{vacuum} + \Delta V_{insulator}$$

$$\Delta V_{vacuum} = \frac{(Q/A)}{\varepsilon_0}(s-d)$$

$$\Delta V_{insulator} = \frac{(Q/A)}{K\varepsilon_0}d$$

$$\Delta V = \frac{(Q/A)}{\varepsilon_0}[s-d(1-1/K)]$$

Biot-Savart Law

Moving charge produces a curly magnetic field

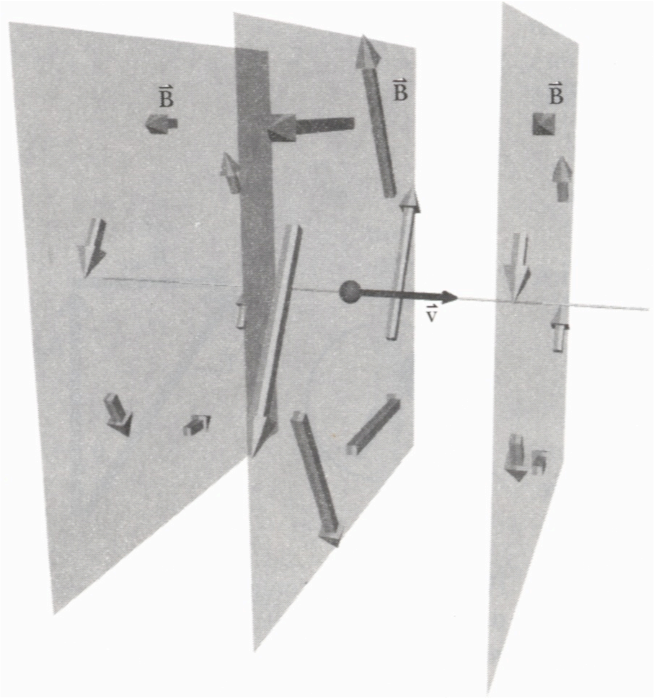
Single Charge:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$$

Current:

$$\Delta\vec{B} = \frac{\mu_0}{4\pi}\frac{I\Delta\vec{l}\times\hat{r}}{r^2}$$

The Biot-Savart law for a short length of thin wire



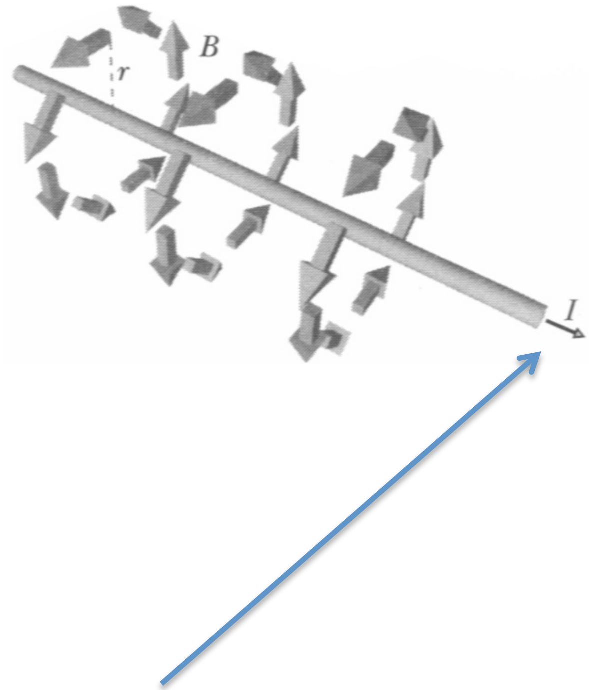
$$I = |q|i = |q|nA\bar{v}$$

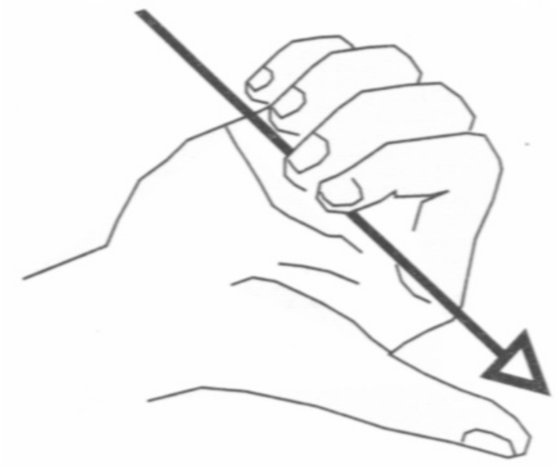
i = electron current = #e/sec *Area * drift velocity

B units: T (Tesla) = kg $s^{-2}A^{-1}$

$$\frac{\mu_0}{4\pi} = 10^{-7}\frac{\mathrm{T\cdot m^2}}{\mathrm{C\cdot m/s}}$$

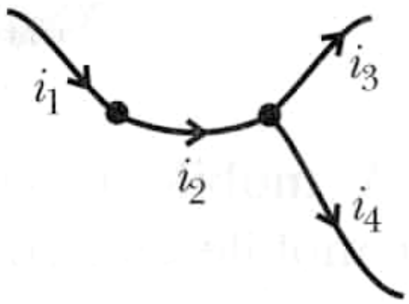
Right-hand Rule for Wire





Conventional Current Direction

Current at a Node

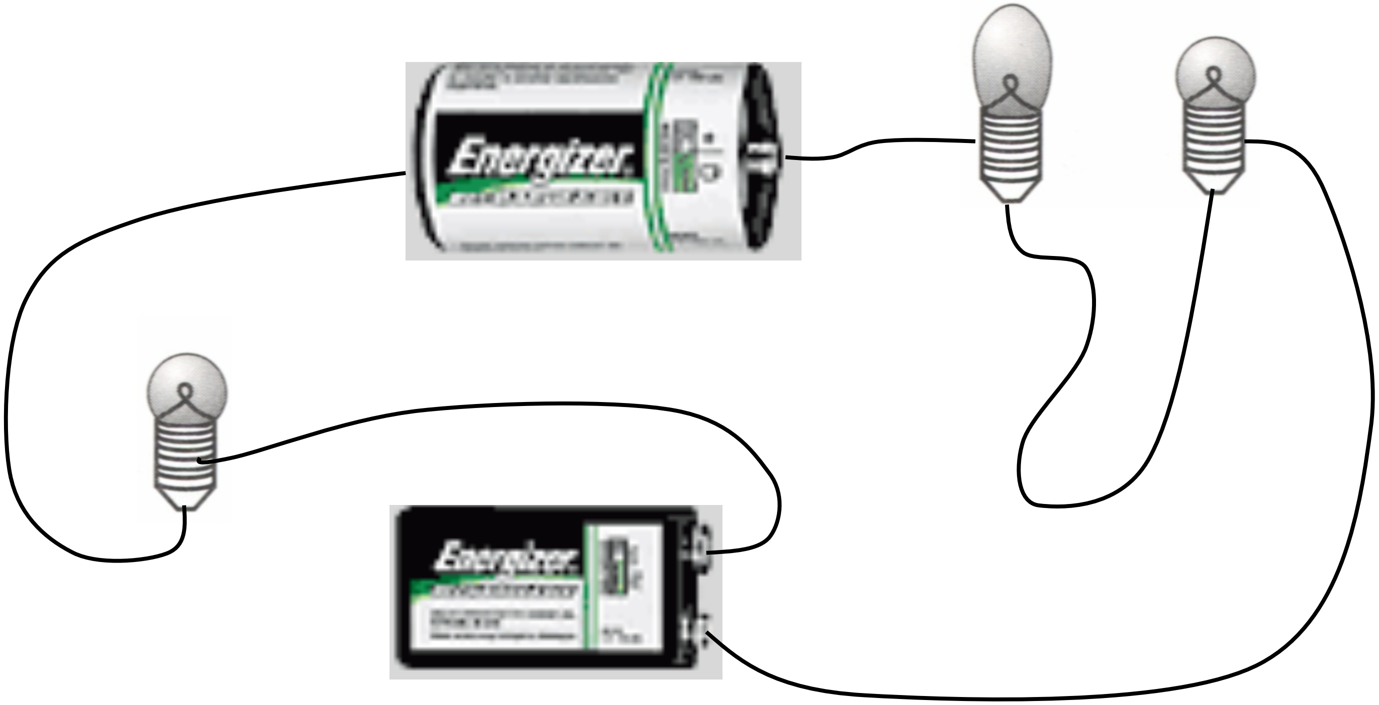


$i_1 = i_2$
$i_2 = i_3 + i_4$

The current node rule
(Kirchhoff node or junction rule [law #1]):

In the steady state, the electron current entering a node in a circuit is equal to the electron current leaving that node

Energy in a Circuit

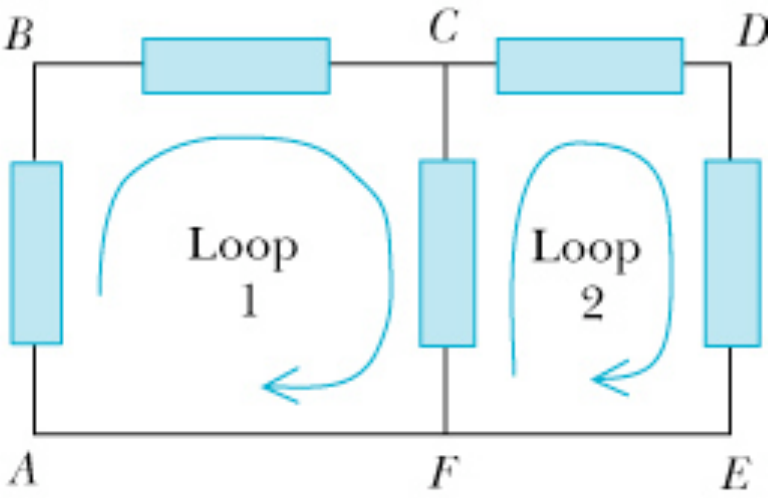


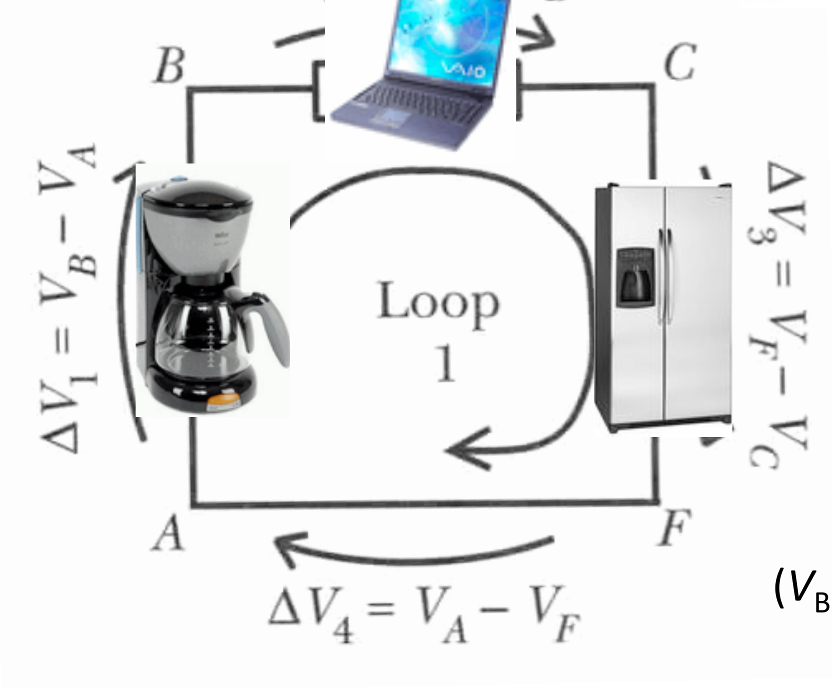
Energy conservation (the Kirchhoff loop rule [2nd law]):

$\Delta V_1 + \Delta V_2 + \Delta V_3 + \ldots = 0$ along any closed path in a circuit

$\Delta V = \Delta U/q$ ← energy per unit charge

General Use of the Loop Rule

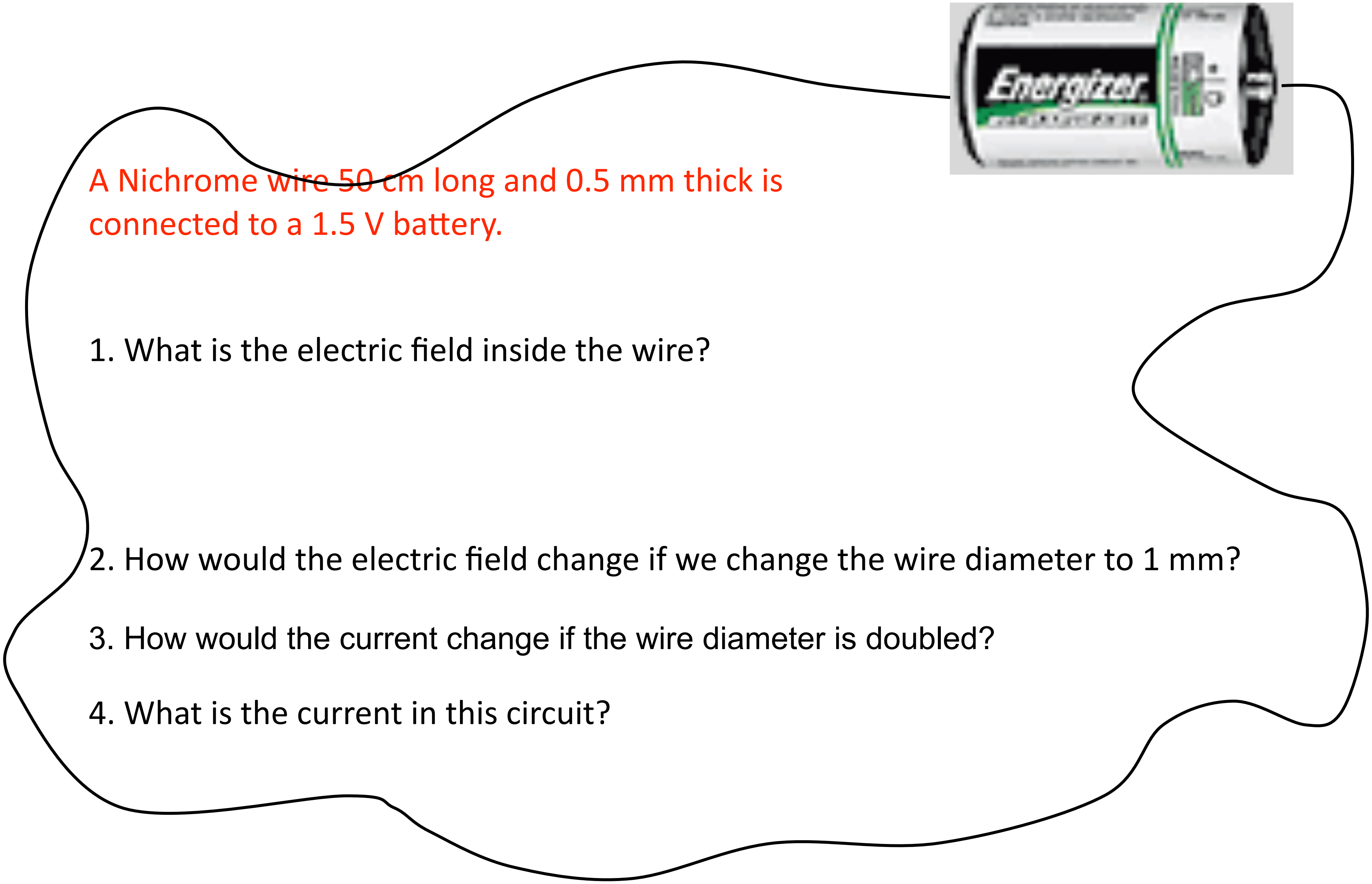




$\Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4 = 0$

$(V_B\text{-}V_A)+ (V_C\text{-}V_B)+ (V_F\text{-}V_C)+ (V_A\text{-}V_F)=0$

Exercise



A Nichrome wire 50 cm long and 0.5 mm thick is connected to a 1.5 V battery.

1. What is the electric field inside the wire?

2. How would the electric field change if we change the wire diameter to 1 mm?

3. How would the current change if the wire diameter is doubled?

4. What is the current in this circuit?

Analysis of Circuits

The current node rule (Charge conservation)

Kirchhoff node or junction rule [1st law]:

In the steady state, the electron current entering a node in a circuit is equal to the electron current leaving that node

Electron current: $i = nAuE$, u = mobility – function of material

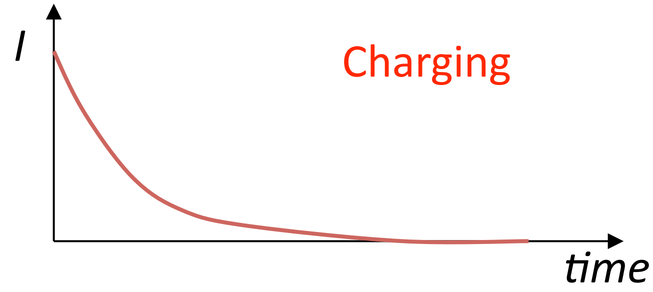
Conventional current: $I = |q|nAuE$

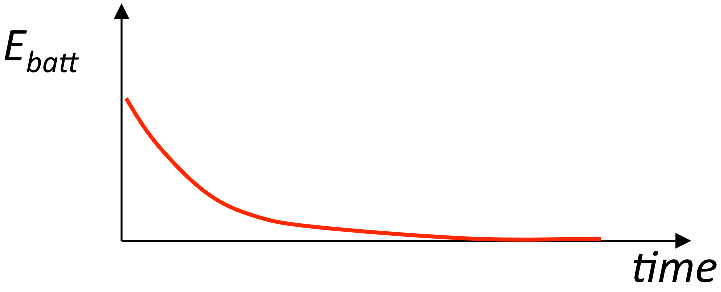
The loop rule (Energy conservation)

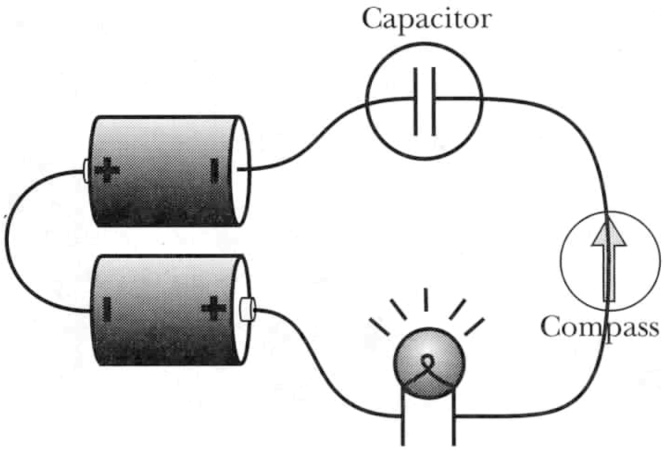
Kirchhoff loop rule [2nd law]:

$\Delta V_1 + \Delta V_2 + \Delta V_3 + \ldots = 0$ along any closed path in a circuit

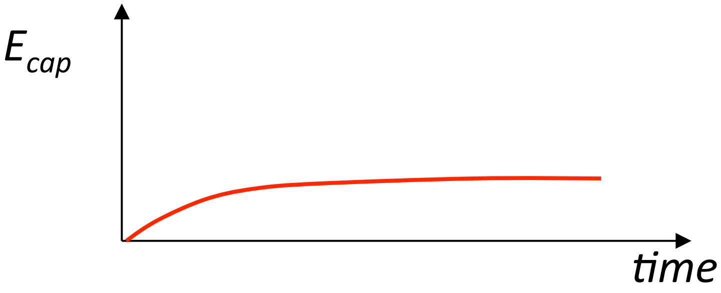
Capacitor in a Circuit





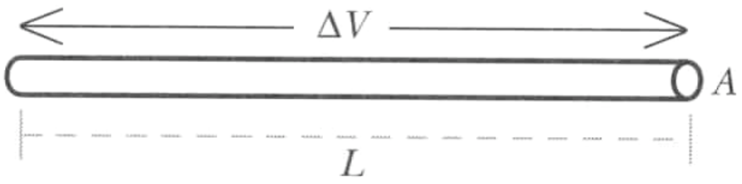


Energy conservation



Resistance

$$\Delta V = -\int_i^f \vec{E} \bullet d\vec{l}$$



$$|\Delta V| = EL \longrightarrow E = \frac{|\Delta V|}{L}$$

$$J = \frac{I}{A} = \sigma E \longrightarrow I = \sigma A E \longrightarrow I = \frac{\sigma A}{L}|\Delta V| = \frac{1}{R}|\Delta V| = \frac{|\Delta V|}{R}$$

$$\boxed{I = \frac{|\Delta V|}{R}} \longleftarrow$$

Widely known as
Ohm’s law

George Ohm
(1789-1854)

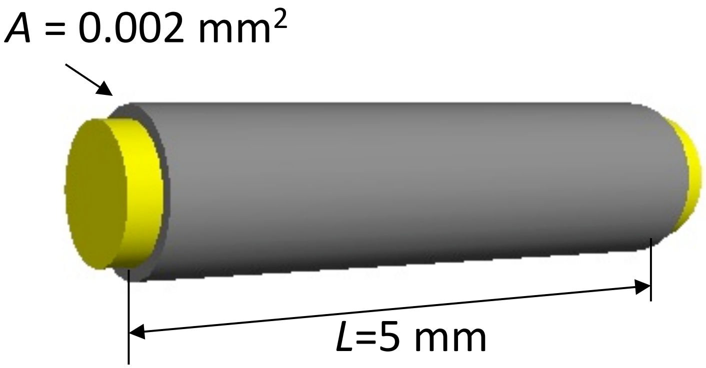
Resistance of a long wire:

$$R = \frac{L}{\sigma A}$$

Units: Ohm, Ω

Resistance combines conductivity and geometry!

Exercise: Carbon Resistor



Conductivity of Carbon:
σ = 3·10^4 (A/m^2)/(V/m)

What is its resistance R?

$$\boxed{R = \frac{L}{\sigma A}}$$

$$R = \frac{(0.005\ \text{m})}{\left(3\times10^4\ \ (\text{A/m}^2)/(\text{V/m})\right)\left(2\times10^{-9}\ \ \text{m}^2\right)} = 83\ \Omega \quad (\text{V/A})$$

What would be the current through this resistor if connected to a 1.5 V battery?

$$\boxed{I = \frac{|\Delta V|}{R}} \longrightarrow I = \frac{1.5\text{V}}{83\Omega} \approx 0.018\ \text{A} = 18\ \text{mA}$$

Ohmic Resistors

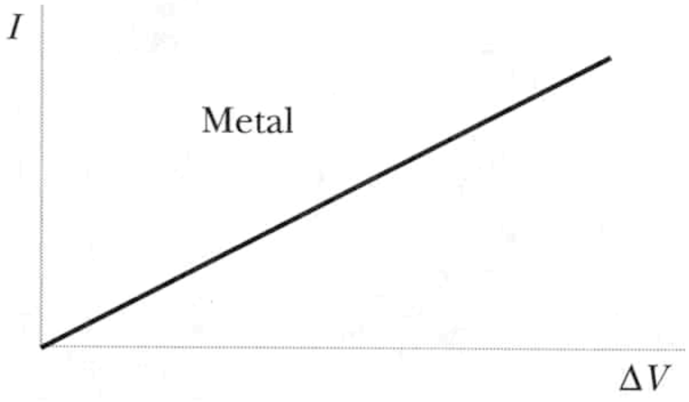
Ohmic resistor: resistor made of ohmic material

Ohmic materials: materials in which conductivity σ is independent of the amount of current flowing through

$$I = \frac{|\Delta V|}{R}$$

$$R = \frac{L}{\sigma A}$$

← not a function of current



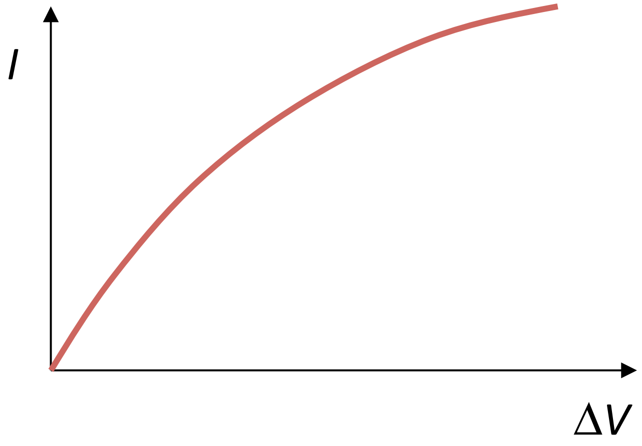
Examples of ohmic materials:
metal, carbon (at constant T!)

Is a Light Bulb an Ohmic Resistor?

Tungsten: mobility at room temperature is larger than at ‘glowing’ temperature (~3000 K)

$$I = \frac{|\Delta V|}{R} \longrightarrow R = \frac{|\Delta V|}{I}$$

V-A dependence:		R
3 V	100 mA	30 Ω
1.5 V	80 mA	19 Ω
0.05 V	6 mA	8 Ω

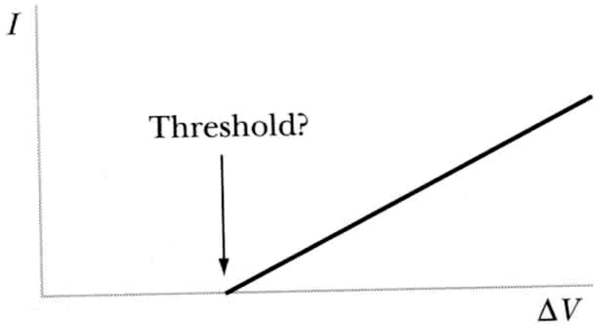


Clearly not ohmic!

Semiconductors

Metals, mobile electrons: slightest ΔV produces current.

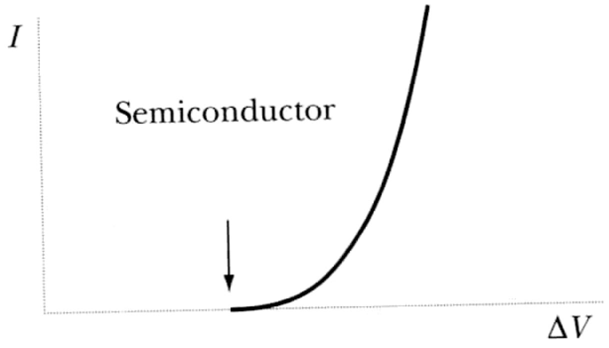
If electrons were bound – we would need to apply some field to free some of them in order for current to flow. Metals do not behave like this!



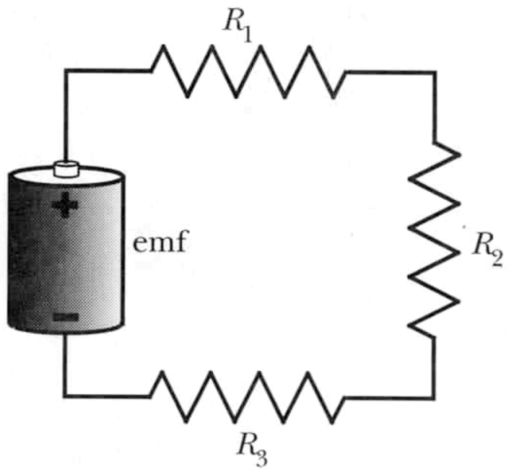
Semiconductors: *n* depends *exponentially* on *E*

$$\sigma = |q|nu \longrightarrow$$ Conductivity depends exponentially on *E*

Conductivity rises (resistance drops) with rising temperature



Series Resistance



$$\Delta V_{\text{batt}} + \Delta V_1 + \Delta V_2 + \Delta V_3 = 0$$

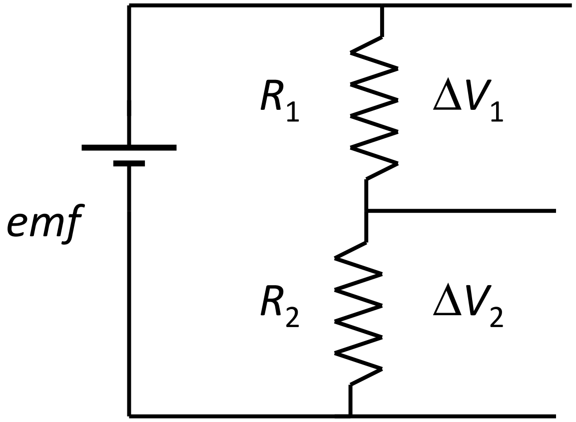
$$emf - R_1 I - R_2 I - R_3 I = 0$$

$$emf = R_1 I + R_2 I + R_3 I$$

$$emf = (R_1 + R_2 + R_3)\, I$$

$$emf = R_{\text{equivalent}}\, I \;, \quad \text{where} \quad \boxed{R_{\text{equivalent}} = R_1 + R_2 + R_3}$$

Exercise: Voltage Divider



Know *R* , find $\Delta V_{1,2}$

Solution:

$$\boxed{I = \frac{|\Delta V|}{R}} \longrightarrow |\Delta V| = IR$$

1) Find current:

$$I = \frac{emf}{R_{equivalent}} = \frac{emf}{R_1 + R_2}$$

2) Find voltage:

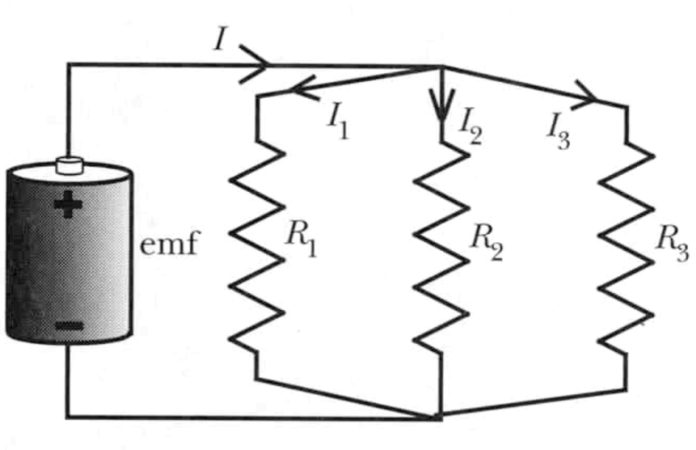
$$|\Delta V_1| = IR_1 = emf\,\frac{R_1}{R_1 + R_2}$$

$$|\Delta V_2| = IR_2 = emf\,\frac{R_2}{R_1 + R_2}$$

3) Check:

$$|\Delta V_1| + |\Delta V_2| = emf \longrightarrow \quad emf\left[\frac{R_1}{R_1 + R_2} + \frac{R_2}{R_1 + R_2}\right] = emf$$

Parallel Resistance



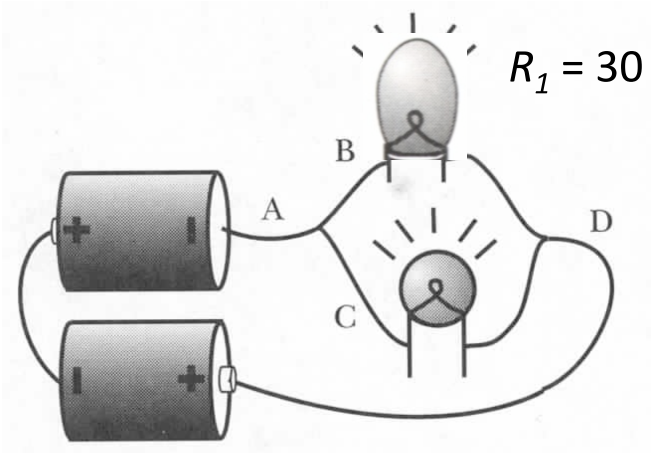
$I = I_1 + I_2 + I_3$

$$I = \frac{emf}{R_1} + \frac{emf}{R_2} + \frac{emf}{R_3}$$

$$I = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) emf = \frac{emf}{R_{equivalent}}$$

$$\boxed{\frac{1}{R_{equivalent}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

Two Light Bulbs in Parallel



R_1 = 30 Ω

R_2 = 10 Ω

What is the equivalent resistance?

$$\boxed{\frac{1}{R_{equivalent}} = \frac{1}{R_1} + \frac{1}{R_2}}$$

$$R_{equivalent} = \frac{R_1 R_2}{R_1 + R_2}$$

$$R_{equivalent} = \frac{300\ \Omega^2}{40\ \Omega} = 7.5\ \Omega$$

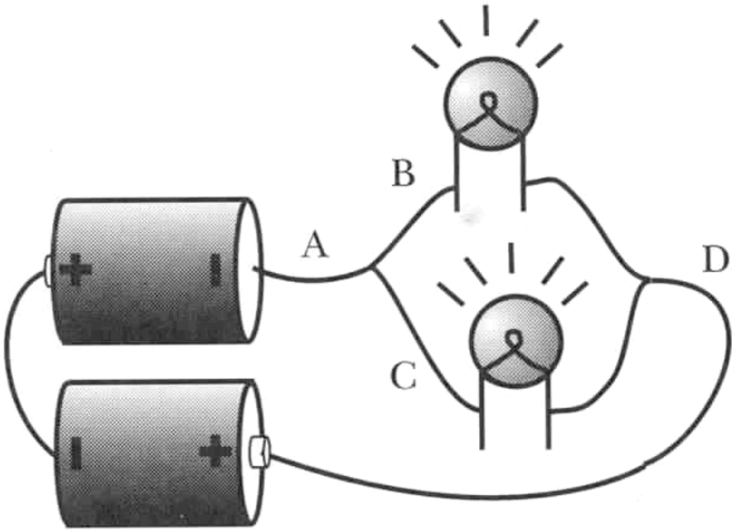
What is the total current?

$$\boxed{I = \frac{|\Delta V|}{R}} = \frac{3\text{ V}}{7.5\ \Omega} = 0.4\text{ A}$$

Alternative way:

$$I = I_1 + I_2 = \frac{|\Delta V|}{R_1} + \frac{|\Delta V|}{R_2} = \frac{3\text{ V}}{30\ \Omega} + \frac{3\text{ V}}{10\ \Omega} = 0.4\text{ A}$$

Two Light Bulbs in Parallel



What would you expect if one is unscrewed?

A) The single bulb is brighter
B) No difference
C) The single bulb is dimmer

Work and Power in a Circuit

Current: charges are moving → work is done

Work = change in electric potential energy of charges

$$\Delta U_e = \Delta q \cdot \Delta V$$

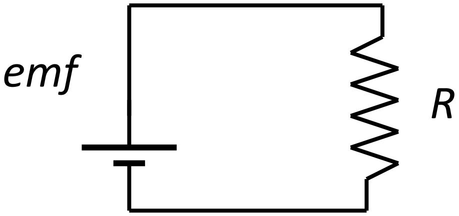
Power = work per unit time:

$$P = \frac{\Delta U_e}{\Delta t} = \frac{\Delta q \cdot \Delta V}{\Delta t} = \underbrace{\frac{\Delta q}{\Delta t}}_{I} \Delta V$$

Power for any kind of circuit component: $P = I\Delta V$

Units: $$\mathrm{AV} = \frac{\mathrm{C}}{\mathrm{s}}\frac{\mathrm{J}}{\mathrm{C}} = \frac{\mathrm{J}}{\mathrm{s}} = \mathrm{W}$$

Power Dissipated by a Resistor



Know ΔV, find P

$$P = I\Delta V$$

$$I = \frac{|\Delta V|}{R}$$

$$\boxed{P = \frac{(\Delta V)^2}{R}}$$

Know I, find P

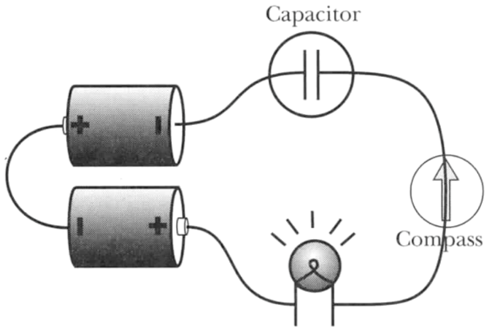
$$P = I\Delta V$$

$$|\Delta V| = IR$$

$$\boxed{P = I^2 R}$$

In practice: need to know *P* to select right size resistor – capable of dissipating thermal energy created by current.

Energy Stored in a Capacitor



$$Q = C\left|\Delta V\right| \longrightarrow \left|\Delta V\right| = \frac{Q}{C}$$

$$dU_{electric} = dQ\Delta V = \frac{Q}{C}dQ$$

$$U_{electric} = \int_0^Q dU_{electric} = \int_0^Q \frac{Q}{C}dQ = \frac{1}{C}\int_0^Q QdQ$$

$$U_{electric} = \frac{1}{2}\frac{Q^2}{C} = \frac{C(\Delta V)^2}{2}$$

Alternative approach:

Energy density: $\frac{\varepsilon_0 E^2}{2}$

$$E = \Delta V / s$$

Energy: $\frac{\varepsilon_0 (\Delta V)^2}{2s^2} \times As = \frac{\varepsilon_0 A (\Delta V)^2}{2s}$

$$C = \frac{\varepsilon_0 A}{s}$$

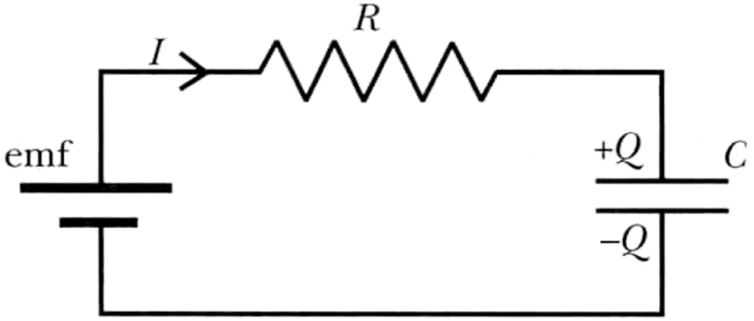
$$= \frac{C(\Delta V)^2}{2}$$

Question

A certain capacitor with only air between its plates has capacitance C and is connected to a battery for a long time until the potential difference across the capacitor is equal to 3 V. The battery is then removed from the circuit and a dielectric (K=2) is inserted between the capacitor plates filling the entire volume. The energy stored in the capacitor With dielectric compare to the energy stored Without dielectric is:

A) The same
B) Larger by a factor of 2
C) [✓ Smaller by a factor of 2]
D) Larger by a factor of 4
E) Smaller by a factor of 4

Quantitative Analysis of an RC Circuit



$$\Delta V_{round\_trip} = emf - RI - \Delta V_C = 0$$

$$\Delta V_C = \frac{Q}{C}$$

$$emf - RI - \frac{Q}{C} = 0$$

$$I = \frac{dQ}{dt} = \frac{emf - Q/C}{R}$$

Initial situation: *Q*=0 $\longrightarrow$ $I_0 = \dfrac{emf}{R}$

Q and *I* are changing in time

$$\frac{d}{dt}$$

$$\frac{dI}{dt} = \frac{d}{dt}\left(\frac{emf}{R}\right) - \frac{d}{dt}\left(\frac{Q}{RC}\right) \longrightarrow \frac{dI}{dt} = -\frac{1}{RC}\frac{dQ}{dt} \longrightarrow \boxed{\frac{dI}{dt} = -\frac{1}{RC}I}$$

RC Circuit: Current

$$\boxed{\frac{dI}{dt} = -\frac{1}{RC} I}$$

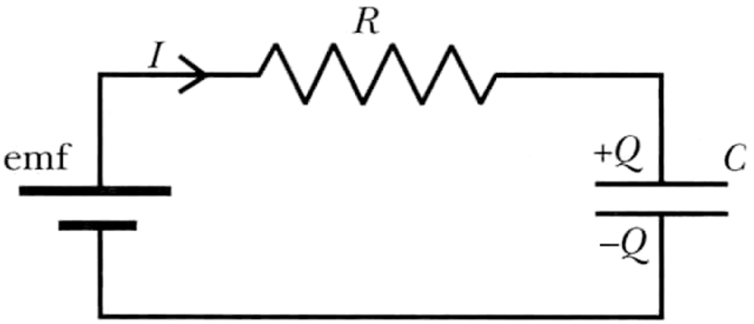
$$\frac{1}{I} dI = -\frac{1}{RC} dt$$

$$\int_{I_0}^{I} \frac{1}{I} dI = -\frac{1}{RC} \int_0^t dt$$

$$\ln I - \ln I_0 = -\frac{t}{RC}$$

$$\ln \frac{I}{I_0} = -\frac{t}{RC}$$

$$\frac{I}{I_0} = e^{-\frac{t}{RC}}$$



Current in an RC circuit

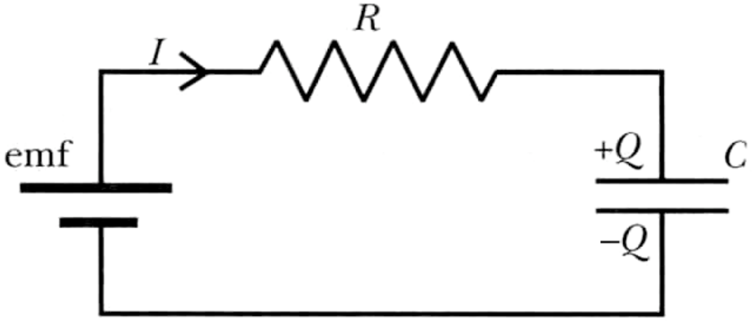
$$I = I_0 e^{-t/RC}$$

What is I_0 ?

Current in an RC circuit

$$I = \frac{emf}{R} e^{-t/RC}$$

RC Circuit: Charge and Voltage



Current in an RC circuit

$$I = I_0 e^{-t/RC}$$

Current in an RC circuit

$$I = \frac{emf}{R} e^{-t/RC}$$

$$I = \frac{dQ}{dt}$$

$$dQ = Idt$$

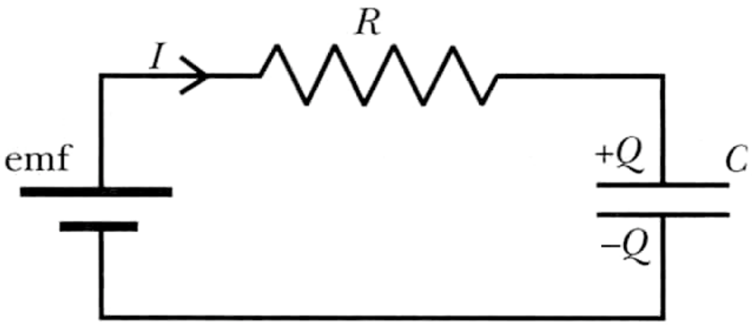
$$Q = \int_0^t Idt = \frac{emf}{R} \int_0^t e^{-t/RC} dt$$

$$Q = C(emf)\left[1 - e^{-t/RC}\right]$$

$$\Delta V = \frac{Q}{C}$$

Check: *t=0, Q=0, t--> inf, Q=C*emf*

RC Circuit: Summary



Current in an RC circuit

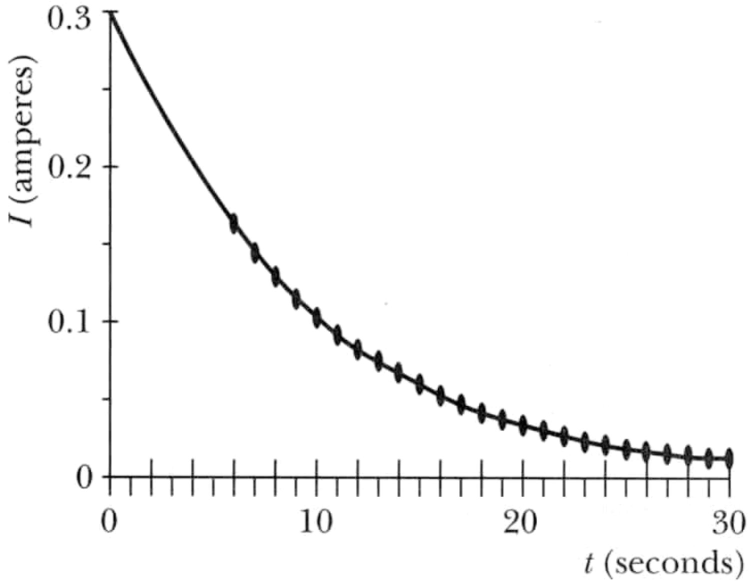
$$I = \frac{emf}{R} e^{-t/RC}$$

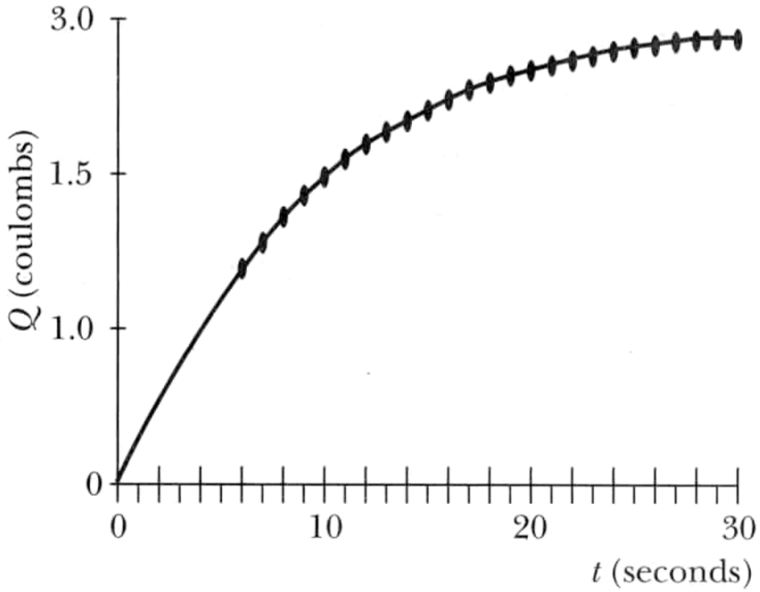
Charge in an RC circuit

$$Q = C\left(emf\right)\left[1 - e^{-t/RC}\right]$$

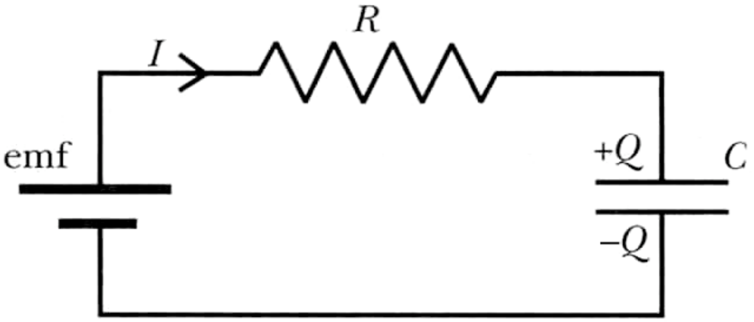
Voltage in an RC circuit

$$\Delta V = \left(emf\right)\left[1 - e^{-t/RC}\right]$$





The RC Time Constant



Current in an RC circuit

$$I = \frac{emf}{R} e^{-t/RC}$$

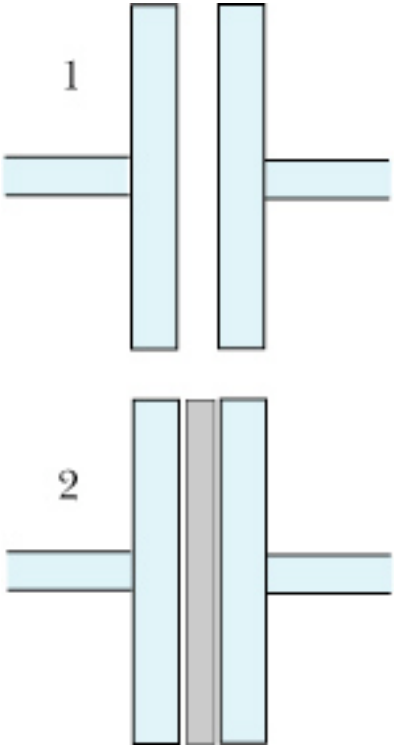
When time *t* = *RC*, the current *I* drops by a factor of *e*.

RC is the ‘time constant’ of an RC circuit.

$$e^{-t/RC} = e^{-1} = \frac{1}{2.718} = 0.37$$

A rough measurement of how long it takes to reach final equilibrium

Consider two capacitors whose only difference is that capacitor number 1 has nothing between the plates, while capacitor number 2 has a layer of plastic in the gap. They are placed in two different circuits having similar batteries and bulbs in series with the capacitor. In the first fraction of a second -

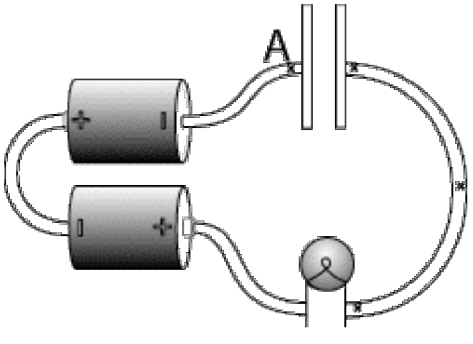


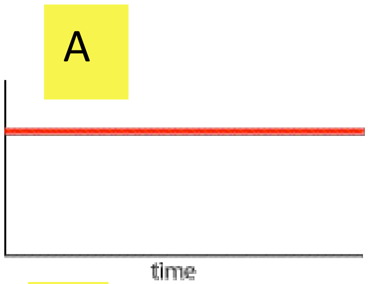
A) The current decreases less rapidly in the circuit containing capacitor 1.

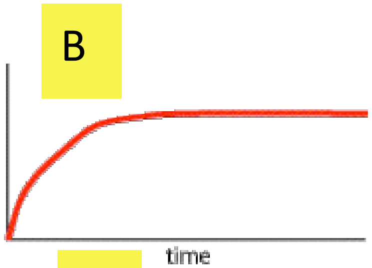
B) The current decreases less rapidly in the circuit containing capacitor 2.

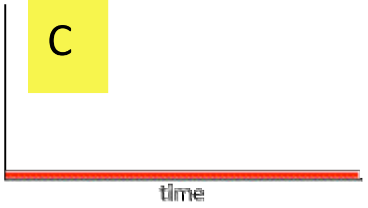
C) The current is the same in both circuits.

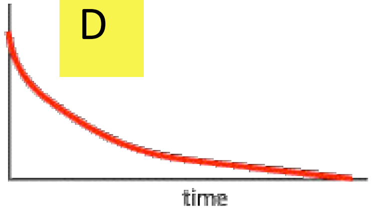
Capacitor initially uncharged. Which graph shows the magnitude of the
POTENTIAL DIFFERENCE across the LIGHT BULB FILAMENT while CHARGING?



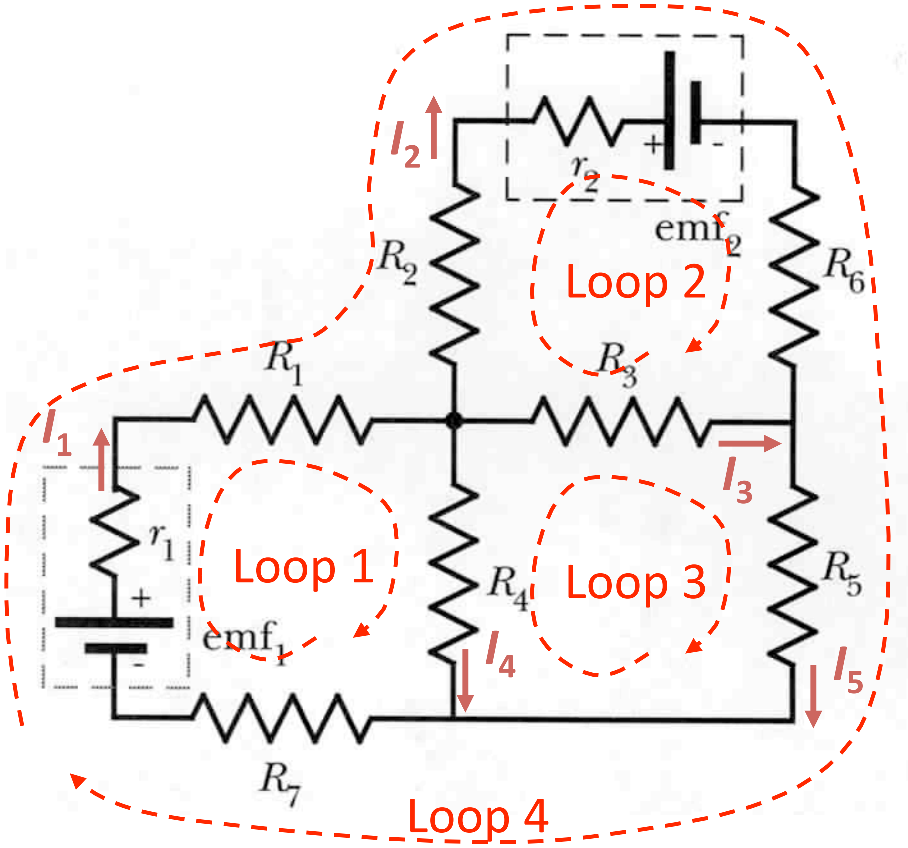








Exercise: A Complicated Resistive Circuit



Find currents through resistors

loop 1:

$$emf - r_1I_1 - R_1I_1 - R_4I_4 - R_7I_1 = 0$$

loop 2:

$$-R_2I_2 - r_2I_2 - emf - R_6I_2 + R_3I_3 = 0$$

loop 3:

$$R_4I_4 - R_3I_3 - R_5I_5 = 0$$

nodes:

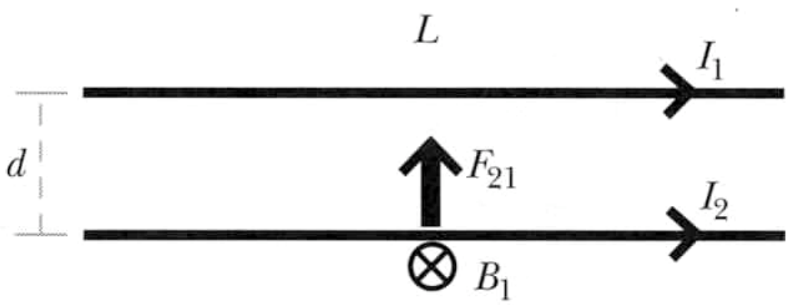
$$I_1 - I_2 - I_3 - I_4 = 0$$

$$I_3 + I_2 - I_5 = 0$$

$$I_4 + I_5 - I_1 = 0$$

Five *independent* equations and five unknowns

Forces Between Parallel Wires



For long wire:

$$B_1 = \frac{\mu_0}{4\pi}\frac{2I_1}{d}$$

Magnetic force on lower wire:

$$\vec{F}_m = I\Delta\vec{l} \times \vec{B}$$

$$\vec{F}_{21} = I_2 L B_1 \sin 90^o$$

$$\boxed{\vec{F}_{21} = I_2 L \frac{\mu_0}{4\pi}\frac{2I_1}{d}}$$

Magnetic force on upper wire:

$$B_2 = \frac{\mu_0}{4\pi}\frac{2I_2}{d}$$

$$\vec{F}_{12} = I_1 L B_2 \sin 90^o$$

$$\vec{F}_{12} = I_1 L \frac{\mu_0}{4\pi}\frac{2I_2}{d}$$

What if current runs in opposite directions?

Electric forces: “likes repel, unlikes attract”

Magnetic forces: “likes attract, unlikes repel”

Gauss’s Law

$$\sum_{surface} \vec{E}\cdot\hat{n}\Delta A = \frac{\sum q_{inside}}{\varepsilon_0}$$

$$\oint \vec{E}\cdot\hat{n}dA = \frac{\sum q_{inside}}{\varepsilon_0}$$

Proof:

1. Proportionality constant
2. Size and shape independence
3. Independence on number of charges inside
4. Charges outside contribute zero

Ampère’s Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \sum I_{inside\_path}$$

All the currents in the universe contribute to *B*
but only ones inside the path result in nonzero path integral

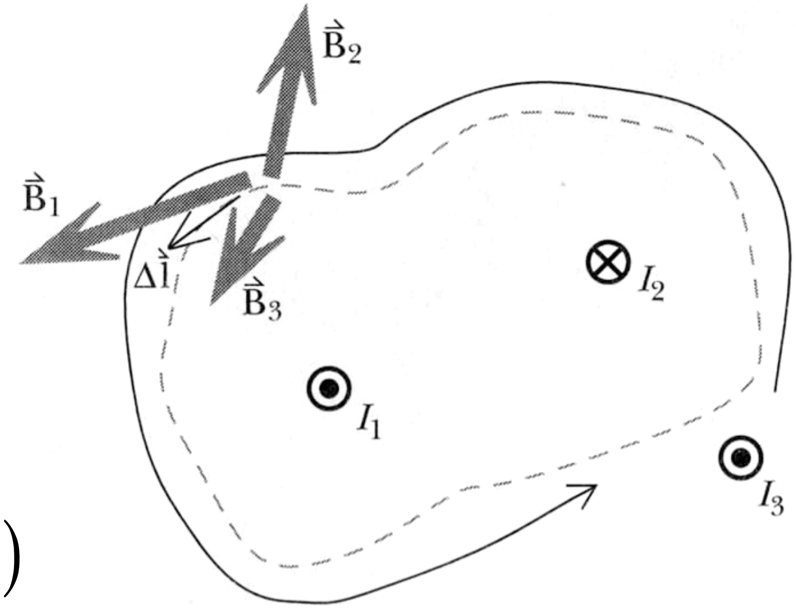
Three Current-Carrying Wires

$$\oint \vec{B}_1 \cdot d\vec{l} = \mu_0 I_1$$

$$\oint \vec{B}_2 \cdot d\vec{l} = -\mu_0 I_2$$

$$\oint \vec{B}_3 \cdot d\vec{l} = 0$$

$$\oint \left(\vec{B}_1 + \vec{B}_2 + \vec{B}_3\right) \cdot d\vec{l} = \mu_0 \left(I_1 - I_2\right)$$



Ampere's law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \sum I_{inside\_path}$$

Faraday’s Law

$$emf = -\frac{d\Phi_{mag}}{dt}$$

Michael Faraday
(1791 - 1867)

Formal version of Faraday’s law:

$$\oint \vec{E}_{NC} \cdot d\vec{l} = -\frac{d}{dt}\left[\int \vec{B} \cdot \hat{n} dA\right]$$

Sign: given by right hand rule

NC = Non-Coulomb

Including Coulomb Electric Field

$$\boxed{\oint \vec{E}_{NC}\cdot d\vec{l} = -\frac{d}{dt}\left[\int \vec{B}\cdot\hat{n}dA\right]}$$

Can we use total *E* in Faraday's law?

$$\oint \vec{E}_{total}\cdot d\vec{l} = \oint\left(\vec{E}_{NC}+\vec{E}_{C}\right)\cdot d\vec{l}$$

$$= \oint \vec{E}_{NC}\cdot d\vec{l} + \underbrace{\oint \vec{E}_{C}\cdot d\vec{l}}_{=0}$$

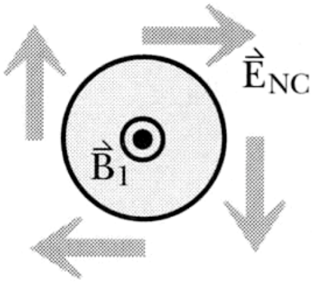
$$\boxed{\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\left[\int \vec{B}\cdot\hat{n}dA\right]}$$

Direction of the Curly Electric Field

Right hand rule:

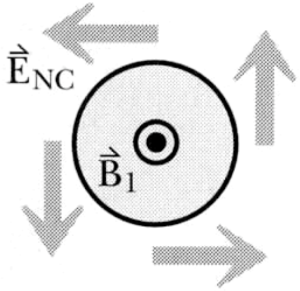
Thumb in direction of $-\frac{d\vec{B}_1}{dt}$

fingers: E_{NC}



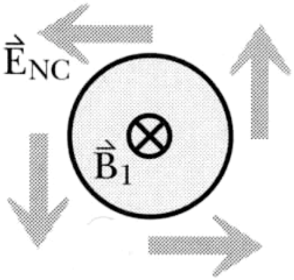
$\vec{B}_1$ out, *increasing*

$-\frac{d\vec{B}_1}{dt}$ into page



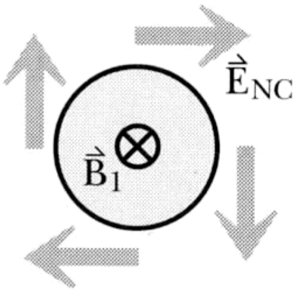
$\vec{B}_1$ out, *decreasing*

$-\frac{d\vec{B}_1}{dt}$ out of page



$\vec{B}_1$ in, *increasing*

$-\frac{d\vec{B}_1}{dt}$ out of page



$\vec{B}_1$ in, *decreasing*

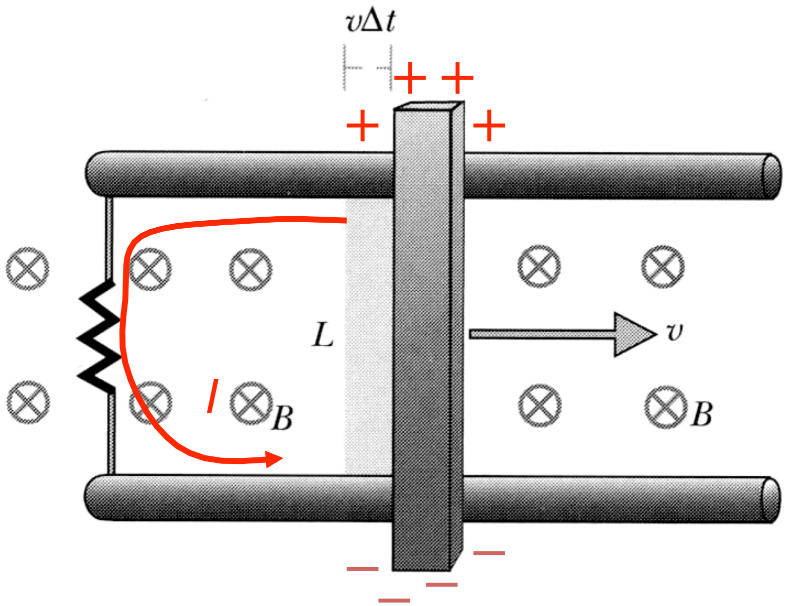
$-\frac{d\vec{B}_1}{dt}$ into page

Faraday’s Law and Motional *EMF*

‘Magnetic force’ approach:

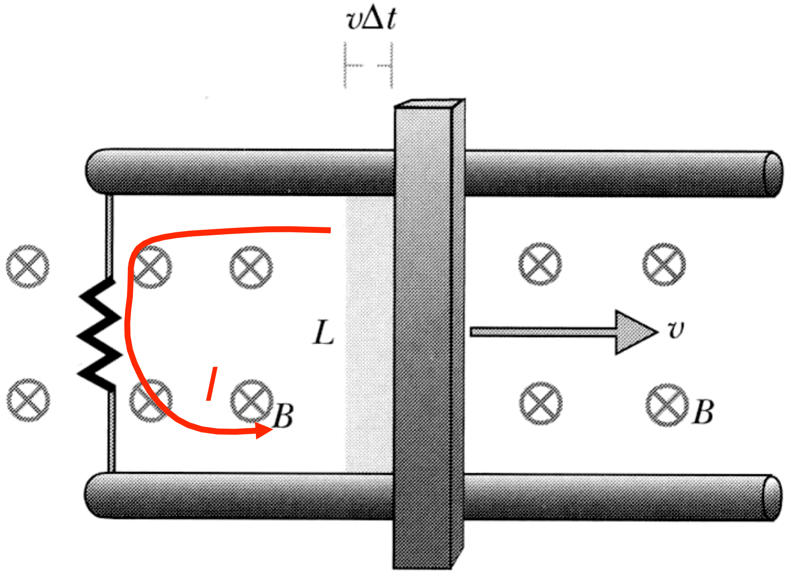
$$\vec{F}_{tot} = q\vec{E} + q\vec{v}\times\vec{B}$$

$$E = vB_\perp \qquad emf = vB_\perp L$$



Use Faraday law:

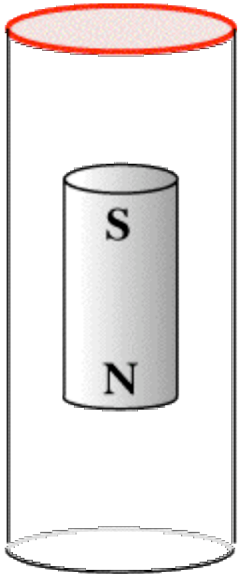
$$\boxed{emf = -\frac{d\Phi_{mag}}{dt}}$$



$$\Delta\Phi_{mag} = B_\perp \Delta A = B_\perp L v\Delta t$$

$$|emf| = \left|\lim_{\Delta t\to 0}\frac{\Delta\Phi_{mag}}{\Delta t}\right| = vB_\perp L$$

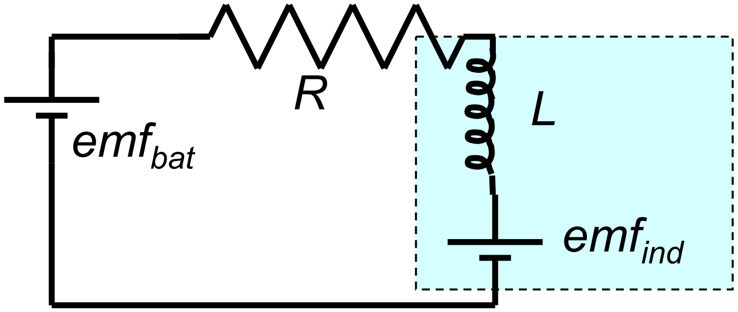
A bar magnet falls through a long aluminum tube.

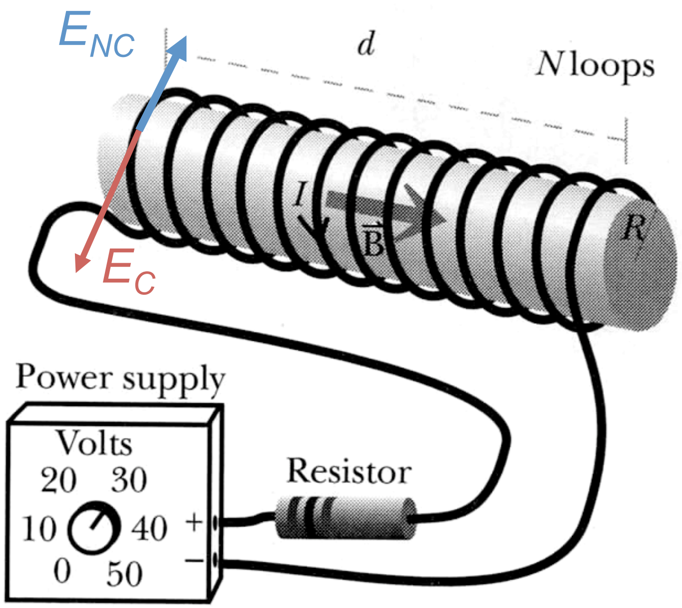


What is the direction of magnetic field at the location of the magnet, due to the current in the red loop?

1) +y (up)
2) –y (down)
3) zero magnitude

Inductance





$$\left|emf_{ind}\right| = L\frac{dI}{dt}$$

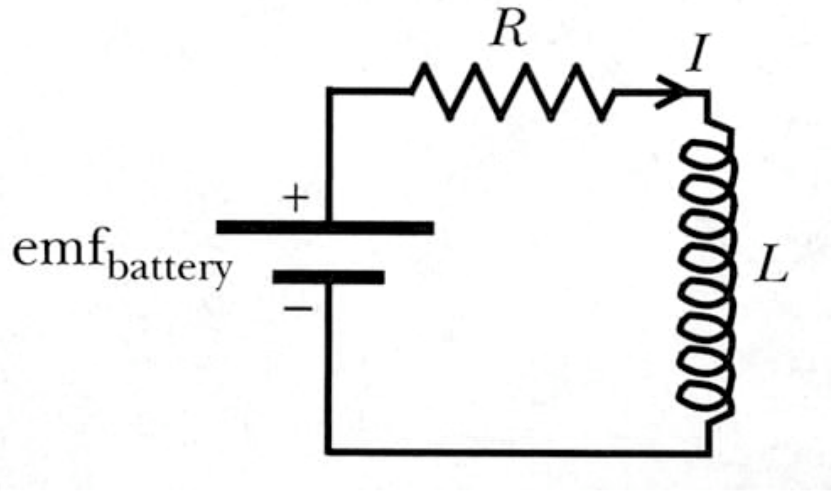
$$\Delta V_{sol} = emf_{ind} - r_{sol}I$$

$$L = \frac{\mu_0 N^2}{d}\pi R^2$$

Unit of inductance *L*:
Henry = Volt·second/Ampere

Increasing the current causes E_{NC} to oppose this increase

Current in RL Circuit



$$\Delta V_{battery} + \Delta V_{resistor} + \Delta V_{inductor} = 0$$

$$emf_{battery} - RI - L\frac{dI}{dt} = 0$$

$$I(t) = a + be^{ct}$$

$$\underbrace{emf_{battery} - Ra}_{\downarrow} - \underbrace{Rbe^{ct} - Lbce^{ct}}_{\downarrow} = 0$$

$$a = \frac{emf_{battery}}{R} \qquad Rb = -Lbc \longrightarrow c = -\frac{R}{L}$$

$$I(t) = \frac{emf_{battery}}{R} + be^{-\frac{R}{L}t}$$

If *t* is very long: $I(t=\infty) = \dfrac{emf_{battery}}{R}$

Current in RL Circuit

$$I(t) = \frac{emf_{battery}}{R} + be^{-\frac{R}{L}t}$$

If *t* is zero: $I(0) = 0$

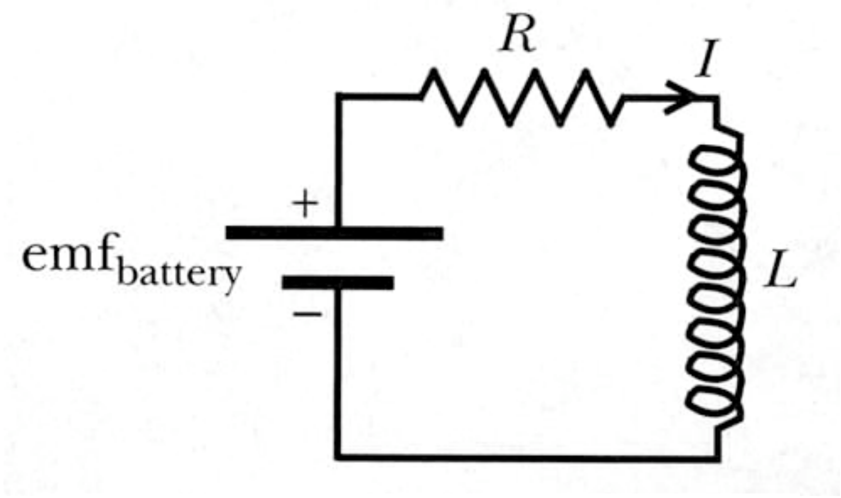
$$I(0) = \frac{emf_{battery}}{R} + b \cdot 1 = 0$$

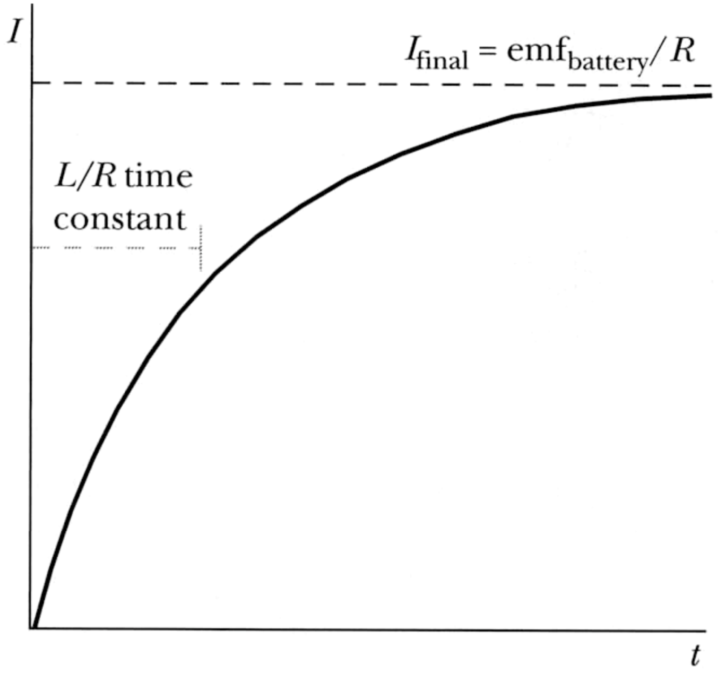
$$\downarrow$$

$$b = -\frac{emf_{battery}}{R}$$

Current in RL circuit:

$$I(t) = \frac{emf_{battery}}{R}\left[1 - e^{-\frac{R}{L}t}\right]$$

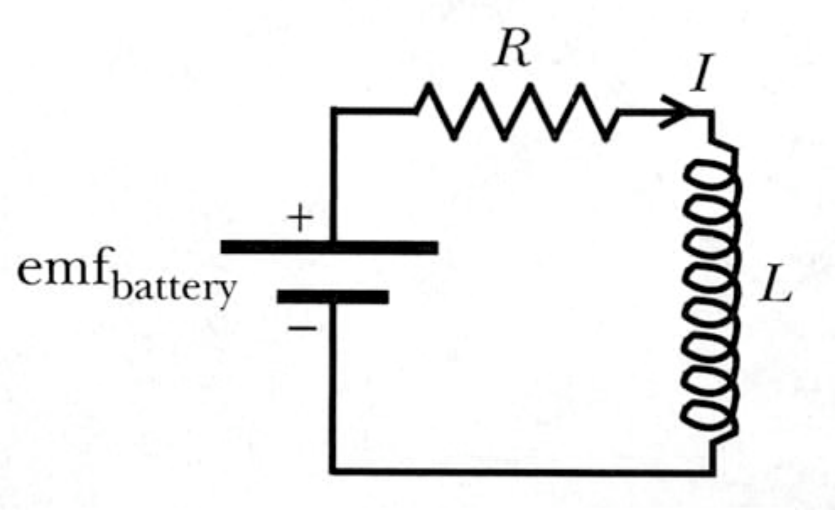




Time Constant of an RL Circuit

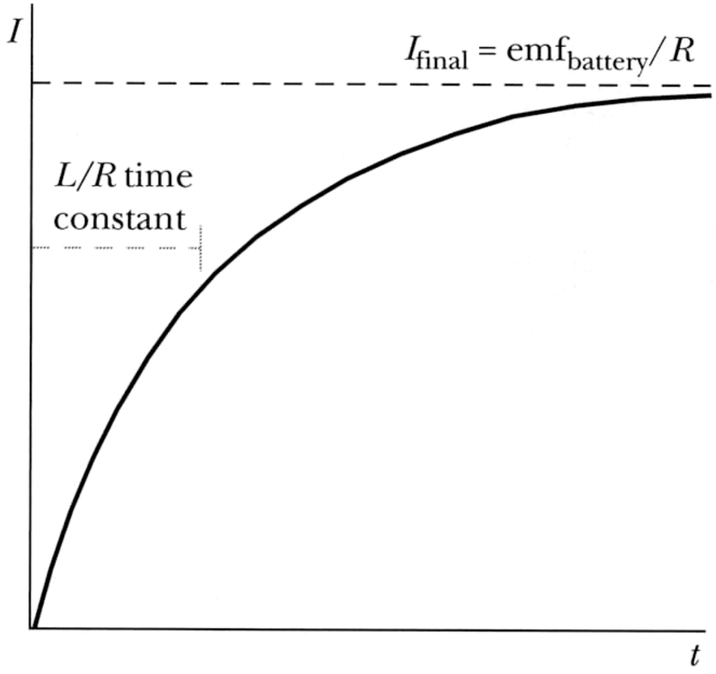
Current in RL circuit:

$$I(t) = \frac{emf_{battery}}{R}\left[1 - e^{-\frac{R}{L}t}\right]$$

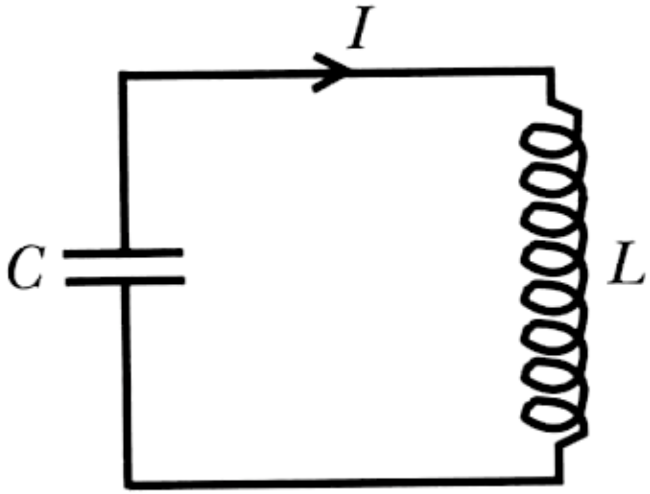


Time constant: time in which exponential factor drops *e* times

$$\frac{R}{L}t = 1 \quad \longrightarrow \quad \tau = \frac{L}{R}$$



Current in an LC Circuit



$$\Delta V_{capacitor} + \Delta V_{inductor} = 0$$

$$\frac{Q}{C} - L\frac{dI}{dt} = 0 \qquad I = -\frac{dQ}{dt}$$

$$Q + LC\frac{d^2Q}{dt^2} = 0$$

$$Q = a + b\cos(ct)$$

$$\underbrace{a}_{a=0} + b\cos(ct) + LC\left(-\underbrace{bc^2}_{c=\frac{1}{\sqrt{LC}}}\cos(ct)\right) = 0$$

$$Q = b\cos\left(\frac{t}{\sqrt{LC}}\right) \longrightarrow \boxed{Q = Q_0\cos\left(\frac{t}{\sqrt{LC}}\right)}$$

Current in an LC Circuit

$$\boxed{Q = Q_0 \cos\left(\frac{t}{\sqrt{LC}}\right)}$$

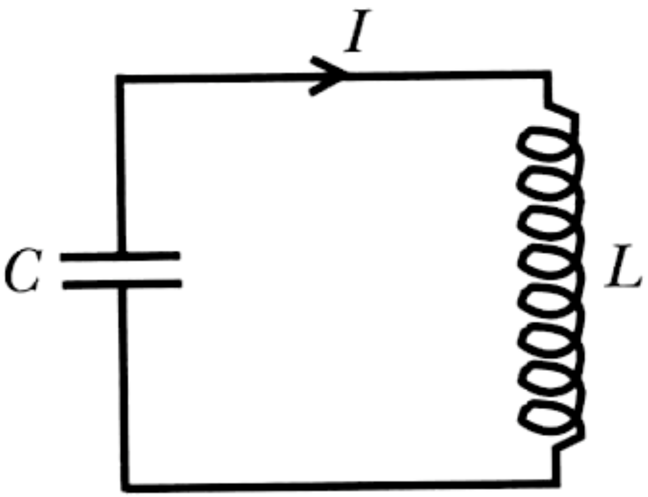
$$I = -\frac{dQ}{dt}$$

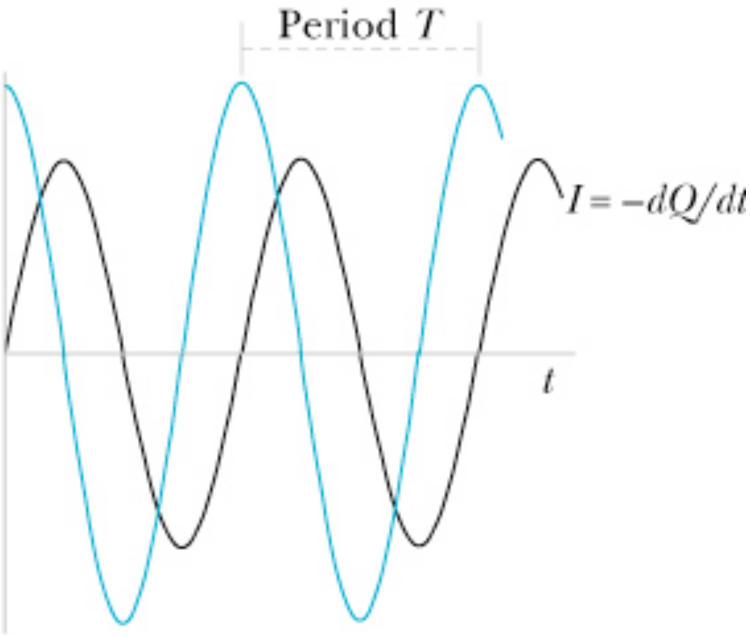
Current in an LC circuit

$$I = \frac{Q_0}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right)$$

Period: $\boxed{T = 2\pi\sqrt{LC}}$

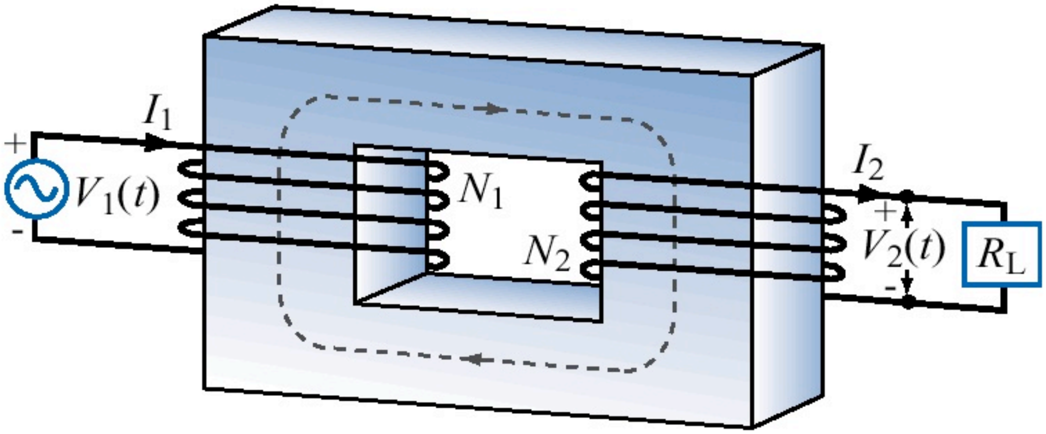
Frequency: $\boxed{f = 1/\left(2\pi\sqrt{LC}\right)}$





Transformer

$$emf_{loop} = -\frac{emf_{AC}}{N_{prim}}$$



$$emf_{\sec} = -\frac{N_{\sec}}{N_{prim}} emf_{AC}$$

Energy conservation:

$$\left|I_{\sec} emf_{\sec}\right| = \left|I_{prim} emf_{AC}\right| \longrightarrow \qquad I_{prim} = -\frac{N_{\sec}}{N_{prim}} I_{\sec}$$