Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Buffer Management for Power Reduction Using Hybrid Control

Jianghai Hu and Yung-Hsiang Lu

Abstract— **This paper proposes a method to manage data buffers for reducing the overall power consumption using the modeling tool of hybrid systems. The problem of designing the most energy-efficient switching strategy for two data buffers inserted between three components in a streamline is formulated as an optimal control problem of a hybrid system. In addition to the control, the optimal shape and size of the state space of the hybrid system also need to be designed at the same time to achieve minimal power consumption. Various necessary conditions of the solutions are presented. In some cases, the optimal switching strategies of the two buffers are derived explicitly.**

I. INTRODUCTION

Conserving the energy of electronic systems has become an increasingly important problem with the increased popularity of battery-powered portable systems. Many methods have been proposed for reducing the energy consumption of individual components in electronic systems, such as processors, wireless network interface cards, or hard disk drives. These methods predict the periods a component will remain idle or under-utilized, and turn it off (sometimes called shut down) or scale down its performance to reduce its energy consumption. This is called the dynamic power management [3].

To ensure smooth system operation, data buffers are often inserted between interacting components of an electronic system. For example, when a user watches a video clip using a PDA (personal digital assistant), the PDA first downloads sufficient amount of data into buffer memory so that the video can be played smoothly regardless of the variations of the network conditions. Using buffers in the dynamic power management makes it possible for under-utilized components to shut down at appropriate times, thus reducing their power consumptions [4], [8]. Fig. 1 illustrates the concept: Component X produces data for Y to consume. Previous studies show how to determine the buffer sizes to achieve maximum power reduction for fixed consumption rates [4] and variable consumption rates [8]. However, they are limited to only one buffer between two components.

J. Hu and Y.-H. Lu are with the School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907, USA. {jianghai,yunglu}@purdue.edu.
Lu is supported in part by NSF Career CNS 0347466.

Fig. 1. A buffer is inserted between two interacting components.

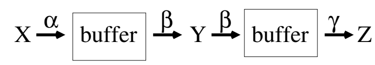


Fig. 2. Two buffers between three interacting components.

In this paper, we study the problem of optimal design and power management of two buffers inserted between three components in a streamline (see Fig. 2) using the modeling framework of hybrid systems. Hybrid systems are proposed to model those systems with both continuous and discrete dynamics, especially those in computer controlled systems. In recent years, hybrid systems are finding increasing applications in all engineering fields. Among the many different formulations of hybrid systems, the one we use here is a version of the piecewise constant hybrid systems, or the so-called *multi-rate automata* [7]. The discrete state has a finite number of possible values (modes), and the continuous state follows constant vector field in each mode. In other words, the continuous state evolves linearly along directions that are dependent on the discrete state.

Among the many aspects of piecewise constant hybrid systems, we focus on their optimal control. We shall model the data flow process in Fig. 2 as a hybrid system, and find the optimal data management strategy by turning on/off components at appropriate time epochs. Compared with the existing literature on optimal control of hybrid systems (e.g, [1], [2], [5], [6], [9], [10], to name a few), a distinguishing feature of our method is that, while trying to find the optimal control strategy, we also study the optimal shape and size of the state space of the hybrid systems *at the same time*. The simultaneous design of the optimal control strategy and the state space makes the problem much more difficult.

II. PROBLEM FORMULATION

A. Data Flow along Three Streamlined Components

Suppose that there are three components in a streamline, denoted as X, Y, and Z. When X is turned on, it

0-7803-9568-9/05/$20.00 ©2005 IEEE

is capable of producing data at the rate of α; when Y is turned on, it is capable of absorbing data from upstream, processing them, and transmitting them downstream, all at the rate of β. Both X and Y can be turned off at any given time, in which case their data rates will drop to zero. In contrast, component Z is assumed to be turned on all the time, and will absorb data from upstream constantly at the rate of γ.

Assume that $\alpha > \gamma$ and $\beta > \gamma$. Thus if both X and Y are turned on all the time, some data will be lost during the transmission process from X to Z. One way to prevent this loss of data is to insert two buffers, one between X and Y, and the other between Y and Z, as is shown in Fig. 2. Data produced by X that is yet to be absorbed by Y can be temporarily stored in buffer 1 whose capacity is denoted as Q_1; data produced by Y that has not been absorbed by Z can be stored in buffer 2 with a capacity Q_2. With the introduction of the two buffers, components X (or Y) can now be turned off occasionally so that Y (or Z) has the time to absorb the data accumulated in buffer 1 (or buffer 2).

To study power consumption, we introduce the following notations. Denote by p_x and p_y the static powers of X and Y, namely, the power consumed by X and Y when they are turned on, respectively. We assume that X and Y consume no power when they are turned off. Denote by q_1 and q_2 the amounts of data stored in buffer 1 and buffer 2 at a given time. Assume that the static powers of buffer 1 and buffer 2 are $p_m^1 Q_1$ and $p_m^2 Q_2$, respectively. Thus the static power consumed by each buffer is proportional to its capacity, and is independent of the actual amount of data stored in it. On the other hand, the dynamical powers of the two buffers, namely, the powers required to keep the stored data refreshed, do depend on the amount of stored data. However, we do not consider the dynamical powers in this paper as it is shown in [4] that they have no effect in the determination of the optimal switching strategy. In addition, we assume that in order to turn X (or Y) from off to on, the extra energy needed is k_x (or k_y). The energy needed to turn X and Y from on to off is assumed to be negligible.

Our goal in this paper is to study the optimal switching strategy for X and Y, in the sense that they are turned on and off at the appropriate time epochs so that data rate as demanded by the component Z is guaranteed, and that the overall system consisting of X, Y, and the two buffers consumes the least amount of average power.

B. Hybrid System Model

We model the data flow process in Fig. 2 as a hybrid system. The discrete state s of the hybrid system has four modes: $S = \{00, 01, 10, 11\}$, where the first and the second bits of s indicate the states of the components X and Y, respectively, with 0 being off and 1 being on. For example, $s = 01$ represents that X is turned off and Y is turned on. The continuous state $q = (q_1, q_2)$ consists of the amounts of data stored in the two buffers, and q takes values in the rectangular state space $Q = [0, Q_1] \times [0, Q_2]$. From the descriptions in the last section, the rates of change for q in the different modes $s \in S$ are: $v_{00} = (0, -\gamma)$, $v_{01} = (-\beta, \beta-\gamma)$, $v_{11} = (\alpha-\beta, \beta-\gamma)$, $v_{10} = (\alpha, -\gamma)$. For example, in state $s = 01$, since X is turned off and Y is turned on, data in buffer 1 will be fetched by Y at the rate of β without any data supply from X, while data in buffer 2 is being fetched by Z at the rate of γ and supplied by Y at the rate of β. Thus $\dot{q}_1 = -\beta$ and $\dot{q}_2 = \beta - \gamma$, rendering $\dot{q} = v_{01} = (-\beta, \beta - \gamma)$. Fig. 3 plots the four vectors v_{00}, v_{01}, v_{11} and v_{10} in the cases of $\alpha > \beta$ and $\alpha < \beta$, respectively.

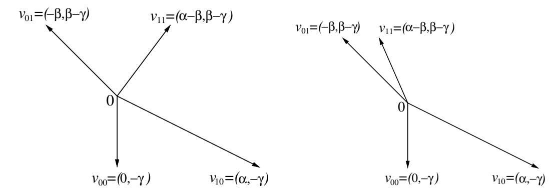


Fig. 3. Control vectors v_s, $s \in S$. Left: $\alpha > \beta$. Right: $\alpha < \beta$.

Given a time period $[0, t_f]$, a switching strategy for the hybrid system is a function $\sigma : [0, t_f] \to S$ so that $\sigma(t)$ represents the states of the two components X and Y at time $t \in [0, t_f]$. Moreover, we assume that there is a partition of $[0, t_f]$, $t_0 = 0 \leq t_1 \leq \ldots \leq t_n = t_f$ for some $n \geq 0$, so that $\sigma(t) \equiv \sigma_i$ is constant on each sub-interval $[t_i, t_{i+1})$, $i = 0, \ldots, n-1$, and that $\sigma(t)$ are different on subsequent sub-intervals. $(\sigma_0, \ldots, \sigma_{n-1})$ is called the switching sequence and $(t_1, \ldots, t_{n-1})$ is called the switching epochs of σ.

Denote by $q(t) = (q_1(t), q_2(t))$ the value of the continuous state, namely, the amounts of data in the two buffers, at time t under strategy σ, starting from the initial value $q(0) = (q_1(0), q_2(0))$. Then

$$\frac{dq(t)}{dt} = v_{\sigma(t)}, \quad \forall t \in [0, t_f]. \tag{1}$$

Together, $z(t) = (q(t), \sigma(t))$ forms the overall trajectory of the system, and is called a *hybrid* trajectory as it includes both continuous and discrete variables.

A *valid* switching strategy is a strategy σ for which the corresponding q belongs to the rectangle $Q = [0, Q_1] \times [0, Q_2]$ at all times so that there is no buffer underflow or overflow. In order for σ to be valid, $v_{\sigma(t)}$ must point toward the inside of Q at any time t where $q(t)$ is on the boundary of Q, which limits the number of possible values for $\sigma(t)$. For example, when $q(t)$ is on the upper boundary of Q, then buffer 2 is full, and Y must be turned off, i.e., $\sigma(t) \in \{00, 10\}$. On the other hand, $\sigma(t)$ can take any of the four values if $q(t)$ is in the interior of Q. We consider only valid strategies in the following, and will simply refer to them as strategies.

C. Energy and Average Power

The dynamic energy of X, Y, and the two buffers does not affect the optimal switching strategies [4]. Hence, we do not consider dynamic energy in this paper.

Let $z(t) = (q(t), \sigma(t))$ be a hybrid trajectory over the time interval $[0, t_f]$. Suppose that the switching strategy σ has switching sequence $(\sigma_0, \ldots, \sigma_{n-1})$ and switching epochs $(t_1, \ldots, t_{n-1})$. Denote by n_x and n_y the numbers of times that X and Y turn from off to on in the switching sequence $(\sigma_0, \ldots, \sigma_{n-1})$, respectively. Denote by $p_{\sigma(t)}$ the instantaneous static power consumed by X and Y together, which, depending on $\sigma(t)$, is given by

$$p_{00} = 0, \quad p_{01} = p_y, \quad p_{10} = p_x, \quad p_{11} = p_x + p_y.$$

The total energy consumed by X, Y and the two buffers during the time period $[0, t_f]$ can then be written as

$$E_\sigma = \int_0^{t_f} p_{\sigma(t)}\, dt + n_x k_x + n_y k_y + (p_m^1 Q_1 + p_m^2 Q_2) t_f$$

Note that E_σ consists of three parts: $\int_0^{t_f} p_{\sigma(t)}\, dt$ is the total static energy consumed by X and Y; $n_x k_x + n_y k_y$ is the total extra energy for switching X and Y from off to on; $(p_m^1 Q_1 + p_m^2 Q_2) t_f$ is the total static energy consumed by the two buffers. Thus the total average power consumed by the whole system during $[0, t_f]$ is

$$\bar{P}(z; t_f, Q_1, Q_2) = \frac{E_\sigma}{t_f}. \tag{2}$$

Problem 1: Find the hybrid trajectories $z(t)$ during $[0, \infty)$ and the buffer sizes Q_1 and Q_2 that minimize $\lim_{t_f \to \infty} \bar{P}(z; t_f, Q_1, Q_2)$.

D. Periodic Hybrid Trajectory

A hybrid trajectory $z(t) = (q(t), \sigma(t))$ over $[0, \infty)$ is called *periodic* with period T if $q(t + T) = q(t)$ and $\sigma(t + T) = \sigma(t)$ for all t. For such $z(t)$, the corresponding strategy σ is uniquely determined by its switching sequence $(\sigma_0, \ldots, \sigma_{n-1})$ and switching epochs $(t_1, \ldots, t_{n-1})$ during the first period $[0, T]$. Note that the periodical condition implies that $\sigma_0 = \sigma_{n-1}$.

For periodical $z(t) = (q(t), \sigma(t))$ with period T, $\lim_{t_f \to \infty} \bar{P}(z; t_f, Q_1, Q_2)$ is equal to the average power consumed during the first period $[0, T]$:

$$\begin{aligned}&\bar{P}(z; T, Q_1, Q_2)\\ &= \frac{1}{T}\left(\sum_{s \in S} T_s p_s + n_x k_x + n_y k_y\right) + p_m^1 Q_1 + p_m^2 Q_2,\end{aligned}$$

where T_s, $s \in S$, is the total amount of time during $[0, T]$ when $\sigma(t) = s$. Note that since $q(T) - q(0) = \sum_{s \in S} T_s v_s$ by (1), and that $q(T) = q(0)$, we must have

Lemma 1 (Constraint on T_s): For periodic $z(t)$, we have $\sum_{s \in S} T_s v_s = 0$.

This establishes two equality constraints for T_s, $s \in S$.

In this paper, we focus on periodic trajectories. Thus Problem 1 reduces to the following problem.

Problem 2: Find a periodical hybrid trajectory $z(t)$ with a proper period T and the proper buffer sizes Q_1 and Q_2 that minimize $\bar{P}(z; T, Q_1, Q_2)$.

III. Necessary Conditions of Solutions

A. Scaling of Hybrid Trajectories

Let $z(t) = (q(t), \sigma(t))$ be a periodic hybrid trajectory with period T for some given Q_1 and Q_2. Thus $q(t)$ is a closed trajectory in $Q = [0, Q_1] \times [0, Q_2]$ satisfying equation (1).

Let λ be a positive number. Then it can be verified that $\lambda q(t/\lambda)$ is a trajectory in $\lambda Q \triangleq [0, \lambda Q_1] \times [0, \lambda Q_2]$ satisfying equation (1) with $\sigma(t)$ replaced by $\sigma(t/\lambda)$. In other words, $(\lambda q(t/\lambda), \sigma(t/\lambda))$ is a periodic hybrid trajectory with the period λT when the maximal sizes of the two buffers become λQ_1 and λQ_2, respectively. We call $z_\lambda(t) \triangleq (\lambda q(t/\lambda), \sigma(t/\lambda))$ the *scaling* of $z(t) = (q(t), \sigma(t))$ by λ. Compared with $z(t)$, the strategy of $z_\lambda(t)$ follows the exact same switching sequence, but the time it spends in each state before switching to a new one is elongated by a factor of λ.

The average power consumed by following $z_\lambda(t)$ is

$$\begin{aligned}&\bar{P}(z_\lambda; \lambda T, \lambda Q_1, \lambda Q_2)\\ &= \tfrac{1}{\lambda T}\left(\textstyle\sum_{s \in S} \lambda T_s p_s + n_x k_x + n_y k_y\right) + p_m^1 \lambda Q_1 + p_m^2 \lambda Q_2\\ &= \lambda(p_m^1 Q_1 + p_m^2 Q_2) + \frac{n_x k_x + n_y k_y}{\lambda T} + \frac{1}{T}\sum_{s \in S} T_s p_s\\ &\geq 2\sqrt{\frac{1}{T}(p_m^1 Q_1 + p_m^2 Q_2)(n_x k_x + n_y k_y)} + \frac{1}{T}\sum_{s \in S} T_s p_s,\end{aligned}$$

where the equality holds if and only if λ takes the value

$$\lambda^* = \sqrt{\frac{n_x k_x + n_y k_y}{(p_m^1 Q_1 + p_m^2 Q_2) T}}. \tag{3}$$

In other words, we have $\bar{P}(\lambda z; \lambda T, \lambda Q_1, \lambda Q_2) \geq \bar{P}(z_{\lambda^*}; \lambda^* T, \lambda^* Q_1, \lambda^* Q_2)$, with equality if and only if $\lambda = \lambda^*$. Let $\lambda = 1$. Then

$$\bar{P}(z; T, Q_1, Q_2) \geq \bar{P}(z_{\lambda^*}; \lambda^* T, \lambda^* Q_1, \lambda^* Q_2), \tag{4}$$

where equality holds if and only if $\lambda^* = 1$.

Now suppose that $z(t)$, Q_1, and Q_2 constitute a solution to Problem 2. Then by (4), we must have $\lambda^* = 1$

for otherwise the new trajectory $z_{\lambda^*}(t)$ will consume strictly less average power than $z(t)$, a contradiction to the assumption that $z(t)$ is optimal. Therefore, we have

Lemma 2 (Constraint on Q_1, Q_2, k_x, k_y and T): A solution $z(t)$, Q_1, and Q_2 to Problem 2 satisfies

$$n_x k_x + n_y k_y = (p_m^1 Q_1 + p_m^2 Q_2)T. \tag{5}$$

Here n_x and n_y are the numbers of times that X and Y turn from off to on in one period, respectively.

In the case that $k_x = k_y = 0$, (5) implies that the optimal buffer sizes are $Q_1 = Q_2 = 0$. In other words, since there is no penalty in turning on and off the two components, one can do so infinitely often so that their effective data rates are equal to γ eventually. On the other hand, if k_x (or k_y) is set very high, then either n_x (or n_y) is zero so that the corresponding component does not switch at all, or in the case of nonzero n_x (or n_y), the optimal buffer size Q_1 (or Q_2) should be made very large, reducing the switching frequency of the corresponding component to very low.

B. Tightness of Optimal $q(t)$ in Q

Note that every bit of buffer space consumes power, which if unused will be a waste. Therefore,

Lemma 3 (Tight in the Box): A solution $z(t) = (q(t), \sigma(t))$, Q_1, and Q_2 to Problem 2 satisfies that

$$\min_{t\in T} q_1(t) = 0, \quad \max_{t\in T} q_1(t) = Q_1,$$
$$\min_{t\in T} q_2(t) = 0, \quad \max_{t\in T} q_2(t) = Q_2.$$

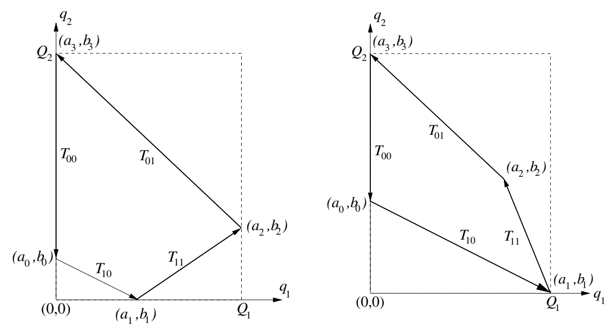
Geometrically, Lemma 3 says that $q(t)$ as a curve in $Q = [0, Q_1] \times [0, Q_2]$ contacts the four edges of Q.

C. Reversing the Switching Sequence

Suppose that $z(t) = (q(t), \sigma(t))$ is a periodic hybrid trajectory for some Q_1 and Q_2 with period T, with switching sequence $(\sigma_0, \ldots, \sigma_{n_1})$ and switching epochs $(t_1, \ldots, t_{n-1})$ in the first period $[0, T]$. We can obtain a new periodic hybrid trajectory with the same buffer sizes Q_1 and Q_2 but a reversed switching sequence $(\sigma_{n-1}, \ldots, \sigma_0)$ in the following way. Define

$$\hat{q}(t) = (Q_1 - q_1(T-t), Q_2 - q_2(T-t)),\ \hat{\sigma}(t) = \sigma(T-t),$$

for $t \in [0, T]$. Thus during $[0, T]$, $\hat{q}(t)$ as a curve in the rectangle Q is obtained by first rotating $q(t)$ around the center $(\frac{Q_1}{2}, \frac{Q_2}{2})$ of Q by 180°, and then reversing its time parameterization. It is easily verified that $\hat{z}(t) = (\hat{q}(t), \hat{\sigma}(t))$ also satisfies equation (1) for $t \in [0, T]$, thus can be extended to a periodic hybrid trajectory on $[0, \infty)$ with period T which for simplicity is also denoted by $\hat{z}(t)$. Note that $z(t)$ and $\hat{z}(t)$ have different starting points. Indeed, $z(0)$ and $\hat{z}(0)$ are symmetric with respect to the center of Q. Moreover,



(a) (b)

Fig. 4. A trajectory $q(t)$ with switching sequence (10,11,01,00). Left: $\alpha > \beta$. Right: $\alpha < \beta$.

Lemma 4: Hybrid trajectories $z(t)$ and $\hat{z}(t)$ have the same average power:

$$\bar{P}(z; T, Q_1, Q_2) = \bar{P}(\hat{z}; T, Q_1, Q_2). \tag{6}$$

IV. Solutions When $n_x = n_y = 1$

In the switching sequence of any periodic hybrid trajectory within its first period, the number of times that X (or Y) switches from off to on is the same as the number of times X (or Y) switches from on to off, and is denoted by n_x (or n_y). We now study the solution $z(t) = (\sigma(t), q(t))$, Q_1, and Q_2 to Problem 2 under the additional constraint that $n_x = n_y = 1$, i.e., X and Y each switches once during one period.

In this case, we need only to consider the switching sequence $(10, 11, 01, 00)$ for $z(t)$. All the other sequences with $n_x = n_y = 1$ can be reduced to this one by one or more of the following operations: (i) Cyclic rotations, such as $(10, 11, 01, 00) \to (11, 01, 00, 10)$, which corresponds to phase shift of the same periodic hybrid trajectory; (ii) Reversions, such as $(10, 11, 01, 00) \to (00, 01, 11, 10)$, described in Section III-C; (iii) Subsequences, for example $(10, 11, 01, 00) \to (10, 11, 00)$, where the latter can be thought of as the degenerate case of the former with the segment representing the state 01 degenerating to zero.

Therefore, we can assume that the solution has the switching sequence $(10, 11, 01, 00)$. Precisely, $q(t)$ starts from an initial position (a_0, b_0) at time 0; travels at the velocity v_{10} for T_{10} time to reach (a_1, b_1); then travels at the velocity v_{11} for T_{11} time to reach (a_2, b_2); then travels at the velocity v_{01} for T_{01} time to reach (a_3, b_3); finally travels at the velocity v_{00} for T_{00} to come back to (a_0, b_0). Overall, $q(t)$ is a periodic trajectory with period $T = T_{00} + T_{10} + T_{11} + T_{01}$. Typical plots of such $q(t)$

can be found in Fig. 4 (a) and (b) for the cases $\alpha > \beta$ and $\alpha < \beta$, respectively.

By Lemma 3, the box $[0, Q_1] \times [0, Q_2]$ should bound $q(t)$ tightly. In the case $\alpha > \beta$, this implies that $a_0 = 0$, $b_1 = 0$, $Q_1 = a_2$ and $Q_2 = b_3$; while in the case $\alpha < \beta$, this implies that $a_0 = 0$, $b_1 = 0$, $Q_1 = a_1$ and $Q_2 = b_3$.

The average power $\bar{P}$ to be minimized is

$$\frac{1}{T}(T_{01}p_{01}+T_{11}p_{11}+T_{10}p_{10}+k_x+k_y)+p_m^1Q_1+p_m^2Q_2.$$

A. The Case $\alpha > \beta$

By Lemma 3, we know that in Fig. 4 (a), $a_0 = 0$, $b_1 = 0$, $a_2 = Q_1$ and $b_3 = Q_2$. In addition, $a_3 = 0$ since the segment corresponding to the state 00 is vertical. We claim that the hybrid trajectory $q(t)$ is completely determined by the two parameters Q_1 and Q_2. In fact, the point $(a_2, b_2) = (Q_1, b_2)$ and T_{01} can be determined from the relation $(Q_1, b_2) = (0, Q_2) - v_{01}T_{01}$ as

$$b_2 = Q_2 - \frac{\beta-\gamma}{\beta}Q_1, \quad T_{01} = \frac{Q_1}{\beta}.$$

Since $(Q_1, b_2) = (a_1, 0) + v_{11}T_{11}$, we deduce that

$$a_1 = \frac{\alpha}{\beta}Q_1 - \frac{\alpha-\beta}{\beta-\gamma}Q_2, \quad T_{11} = \frac{Q_2}{\beta-\gamma} - \frac{Q_1}{\beta}.$$

Then, from $(a_1, 0) = (0, b_0) + v_{10}T_{10}$, we have

$$b_0 = \frac{\gamma}{\beta}Q_1 - \frac{\gamma(\alpha-\beta)}{\alpha(\beta-\gamma)}Q_2, \ \ T_{10} = \frac{Q_1}{\beta} - \frac{(\alpha-\beta)Q_2}{\alpha(\beta-\gamma)}.$$

Note that in order to have $T_{10} \geq 0$, T_1 should satisfy

$$Q_1 \geq \frac{\beta(\alpha-\beta)}{\alpha(\beta-\gamma)}Q_2. \tag{7}$$

Finally, from $(0, b_0) = (0, Q_2) + v_{00}T_{00}$, we have

$$T_{00} = -\frac{Q_1}{\beta} + \frac{\beta(\alpha-\gamma)Q_2}{\alpha\gamma(\beta-\gamma)}.$$

As a result, $T = T_{00} + T_{01} + T_{11} + T_{10}$ is given by $T = \frac{\beta}{\gamma(\beta-\gamma)}Q_2$. The average power is then reduced to

$$\bar{P} = p_m^1Q_1 + p_m^2Q_2 + \frac{C_1}{Q_2} + C_2, \tag{8}$$

where C_1 and C_2 are positive constants defined by

$$C_1 = \frac{\gamma(\beta-\gamma)(k_x+k_y)}{\beta}, \quad C_2 = \frac{\gamma}{\alpha}p_x + \frac{\gamma}{\beta}p_y. \tag{9}$$

To find the optimal values Q_1^* and Q_2^* that minimize $\bar{P}$, first note that for each fixed Q_2, $\bar{P}$ is minimized by the smallest possible value of Q_1, which is given by the right hand side of (7). Thus

$$\min_{Q_1,Q_2} \bar{P} = \min_{Q_2}\min_{Q_1} \bar{P}$$
$$= \min_{Q_2}\left\{\left[\frac{\beta(\alpha-\beta)}{\alpha(\beta-\gamma)}p_m^1 + p_m^2\right]Q_2 + \frac{C_1}{Q_2} + C_2\right\},$$

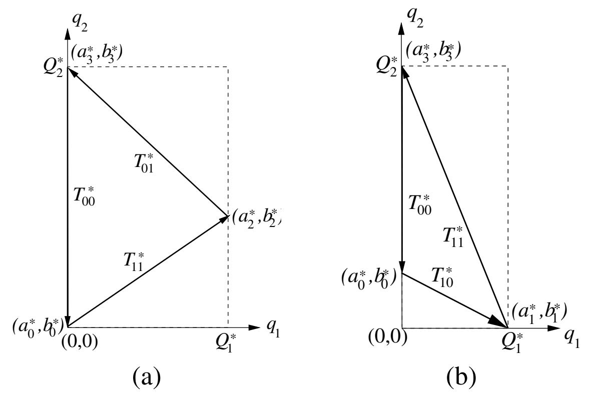


Fig. 5. Optimal trajectory $q^*(t)$. Left: $\alpha > \beta$. Right: $\alpha < \beta$.

which achieves its minimum at

$$Q_2^* = (\beta-\gamma)\sqrt{\frac{\alpha\gamma(k_x+k_y)}{\beta^2(\alpha-\beta)p_m^1 + \alpha\beta(\beta-\gamma)p_m^2}}. \tag{10}$$

Hence the optimal Q_1^* is

$$Q_1^* = \frac{\beta(\alpha-\beta)}{\alpha(\beta-\gamma)}Q_2^*$$
$$= (\alpha-\beta)\sqrt{\frac{\beta\gamma(k_x+k_y)}{\alpha\beta(\alpha-\beta)p_m^1 + \alpha^2(\beta-\gamma)p_m^2}}. \tag{11}$$

The corresponding minimum average power $\bar{P}^*$ is

$$\bar{P}^* = 2\sqrt{\frac{\gamma}{\alpha\beta}(k_x+k_y)\left[\beta(\alpha-\beta)p_m^1 + \alpha(\beta-\gamma)p_m^2\right]}$$
$$+ \frac{\gamma}{\alpha}p_x + \frac{\gamma}{\beta}p_y, \tag{12}$$

and the optimal period T^* is

$$T^* = \sqrt{\frac{\alpha\beta(k_x+k_y)}{\beta\gamma(\alpha-\beta)p_m^1 + \alpha\gamma(\beta-\gamma)p_m^2}}. \tag{13}$$

Since the optimal Q_1^* and Q_2^* is such that equality holds in (7), we conclude that $T_{10}^* = 0$, i.e., in the optimal solution q^*, the segment corresponding to the state $s = 10$ degenerates to a single point, namely, the origin. See Fig. 5 (a) for a plot of q^*.

Intuitively, by Fig. 5 (a), the optimal solution will start at time 0 with both buffers empty and both components X and Y turning on. Since $\alpha > \beta > \gamma$, data will accumulate in both buffers. After T_{11}^* time, buffer 1 will reach its capacity Q_1^* first. Then X switches off, and the data in buffer 1 starts decreasing while data in buffer 2 keeps accumulating. After T_{01}^* time, buffer 1 is again emptied and buffer 2 also reaches its maximal capacity Q_2^* at the same time. Then Y switches off, and data in buffer 2 are gradually fetched by Z until it is empty. This whole process is then repeated every T^* time.

B. The Case $\alpha < \beta$

In this case, we have $a_0 = 0$, $b_1 = 0$, $Q_1 = a_1$ and $Q_2 = b_3$, and the trajectory $q(t)$ as plotted in Fig. 4 (b) is fully determined by the two parameters Q_1 and Q_2. Indeed, the vertex $(a_0, b_0) = (0, b_0)$ can be solved from the relation $(0, b_0) = (Q_1, 0) - T_{10}v_{10} = (0, Q_2) + T_{00}v_{00}$ as $b_0 = \frac{\gamma}{\alpha}Q_1, T_{00} = \frac{Q_2}{\gamma} - \frac{Q_1}{\alpha}$, and $T_{10} = \frac{Q_1}{\alpha}$. On the other hand, solving $(a_2, b_2) = (Q_1, 0) + T_{11}v_{11} = (0, Q_2) - T_{01}v_{01}$ yields

$$a_2 = \frac{\beta}{\alpha}Q_1 + \frac{\beta(\alpha - \beta)}{\alpha(\beta - \gamma)}Q_2,\ b_2 = -\frac{\beta - \gamma}{\alpha}Q_1 + \frac{\beta}{\alpha}Q_2,$$

$$T_{01} = \frac{Q_1}{\alpha} + \frac{(\alpha - \beta)Q_2}{\alpha(\beta - \gamma)},\ T_{11} = -\frac{Q_1}{\alpha} + \frac{\beta Q_2}{\alpha(\beta - \gamma)}.$$

Note that, since $T_{01} \geq 0$, we must have

$$Q_1 \geq \frac{\beta - \alpha}{\beta - \gamma}Q_2. \qquad (14)$$

The period of $q(t)$ is $T = T_{00} + T_{11} + T_{01} + T_{10} = \frac{\beta}{(\beta - \gamma)\gamma}Q_2$. Thus the average power is reduced to

$$\bar{P} = p_m^1 Q_1 + p_m^2 Q_2 + \frac{C_1}{Q_2} + C_2, \qquad (15)$$

where C_1 and C_2 are constants defined in (9).

Note that $\bar{P}$ has the exact same expression as in the previous case, despite the different expressions for $T_{00}, T_{01}, T_{10}, T_{11}$, etc. Therefore, we can follow similar steps to find the optimal Q_1^* and Q_2^* that minimize $\bar{P}$. We omit the detailed derivation, and write the results as:

$$Q_1^* = (\beta - \alpha)\sqrt{\frac{\gamma(k_x + k_y)}{\beta(\beta - \alpha)p_m^1 + \beta(\beta - \gamma)p_m^2}}. \qquad (16)$$

$$Q_2^* = (\beta - \gamma)\sqrt{\frac{\gamma(k_x + k_y)}{\beta(\beta - \alpha)p_m^1 + \beta(\beta - \gamma)p_m^2}}. \qquad (17)$$

The corresponding average power $\bar{P}^*$ and period T^* are

$$\bar{P}^* = 2\sqrt{\frac{\gamma}{\beta}(k_x + k_y)\left[(\beta - \alpha)p_m^1 + (\beta - \gamma)p_m^2\right]} + \frac{\gamma}{\alpha}p_x + \frac{\gamma}{\beta}p_y, \qquad (18)$$

$$T^* = \sqrt{\frac{\beta(k_x + k_y)}{\gamma(\beta - \alpha)p_m^1 + \gamma(\beta - \gamma)p_m^2}}. \qquad (19)$$

Since equality holds in (14), in Fig. 4 (b), the segment corresponding to the state $s = 01$ degenerates into a single point $(0, Q_2^*)$. See Fig. 5 (b) for a plot of the corresponding optimal solution q^*.

According to q^*, at time 0, buffer 1 is empty while buffer 2 is partially filled; component X is on and component Y is off. Thus data accumulate in buffer 1 and decrease in buffer 2. After exactly T_{10}^* time, buffer 1 reaches its capacity Q_1^* and buffer 2 is emptied simultaneously. Then component Y is switched on. Since $\alpha < \beta$, data in buffer 1 will start decreasing, and data in buffer 2 will start increasing as $\beta > \gamma$. After exactly T_{11}^* time, buffer 1 is emptied again while buffer 2 reaches its capacity Q_2^* simultaneously. At this time, both X and Y switch off. Data in buffer 2 will be gradually fetched by Z until the amount of data left becomes b_0^* after T_{00}^* time. This process is then repeated every T^* time.

V. Conclusion

In this paper we study the problem of power management for three components with two buffers in between. Optimal buffer sizes and switching strategies are derived analytically in the case when each buffer is allowed at most one switching in a cycle.

The problem studied in this paper can be generalized in several ways. To name a few, the buffers can have multiple switching in a single cycle; the number of buffers (components) can be larger than two (three); the rates of the components can be variable or even random instead of constant. To study the optimal buffer sizes and switching strategies in these cases will be our future direction of research.

References

[1] R. Alur, S. La Torre, and G. Pappas. Optimal paths in weighted timed automata. In M.D. Di Benedetto and A. Sangiovanni-Vincentelli, editors, *Hybrid Systems: Computation and Control, 4th Int. Workshop*, volume 2034 of *Lecture Notes in Computer Science*, pages 49–62. Springer-Verlag, Rome, Italy, 2001.

[2] M. Baotić, F. J. Christophersen, and M. Morari. A new algorithm for constrained finite time optimal control of hybrid systems with a linear performance index. In *Proc. of the European Control Conference*, Cambridge, UK, September 2003.

[3] L. Benini and G. De Micheli. System-Level Power Optimization: Techniques and Tools. *ACM Transactions on Design Automation of Electronic Systems*, 5(2):115–192, April 2000.

[4] L. Cai and Yung-Hsiang Lu. Dynamic Power Management Using Data Buffers. In *Design Automation and Test in Europe*, pages 526–531, 2004.

[5] M. Egerstedt, Y. Wardi, and F. Delmotte. Optimal control of switching times in switched dynamical systems. In *Proc. 42nd IEEE Int. Conf. on Decision and Control*, volume 3, pages 2138–2143, Maui, HI, Dec. 2003.

[6] S. Hedlund and A. Rantzer. Optimal control of hybrid systems. In *Proc. 38th IEEE Int. Conf. on Decision and Control*, pages 3972–3977, Phoenex, AZ, Dec. 1999.

[7] T. A . Henzinger, P. W. Kopke, A. Puri, and P. Varaiya. What's decidable about hybrid automata? In *Proceedings of STOC: Theory of Computing*, pages 373–382. ACM Press, 1995.

[8] N. Pettis, L. Cai, and Y.-H. Lu. Dynamic Power Management for Streaming Data. In *International Symposium on Low Power Electronics and Design*, pages 62–65, 2004.

[9] H. J. Sussmann. A maximum principle for hybrid optimal control problems. In *Proc. 38th IEEE Int. Conf. on Decision and Control*, pages 425–430, Phoenex, AZ, Dec. 1999.

[10] X. Xu and P.J. Antsaklis. Optimal control of switched systems based on parameterization of the switching instants. *IEEE Trans. Automatic Control*, 49(1):2–16, Jan. 2004.