# Quantum Phase Estimation: High-Dimensional Photonic Advancements

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## Core Concepts

### What is a quantum phase estimation algorithm (PEA)?

A quantum phase estimation algorithm is a means of determining an unknown eigenphase of a unitary matrix $U$,

$$U|\nu_k\rangle = e^{i2\pi\theta_k}|\nu_k\rangle.$$

where $|\nu_k\rangle$ is any eigenstate of $U$, $e^{i2\pi\theta_k}$ is its corresponding eigenvalue and $\theta_k$ is called the eigenphase. While $U$ can be *applied* to quantum states, it is treated as an unknown black box[1].

### Why PEA?

Quantum phase estimation algorithms (PEAs) are is key subroutine in many quantum algorithms, including:

- Shor's factoring algorithm[2]
- Quantum principal component analysis[3]
- Generalized Grover's search algorithm[4]
- Quantum simulations[5,6]


[1] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2011. isbn: 1107002176

[2] P. W. Shor. "Algorithms for quantum computation: discrete logarithms and factoring". *Proceedings 35th annual symposium on foundations of computer science.* IEEE. 1994, pp. 124–134

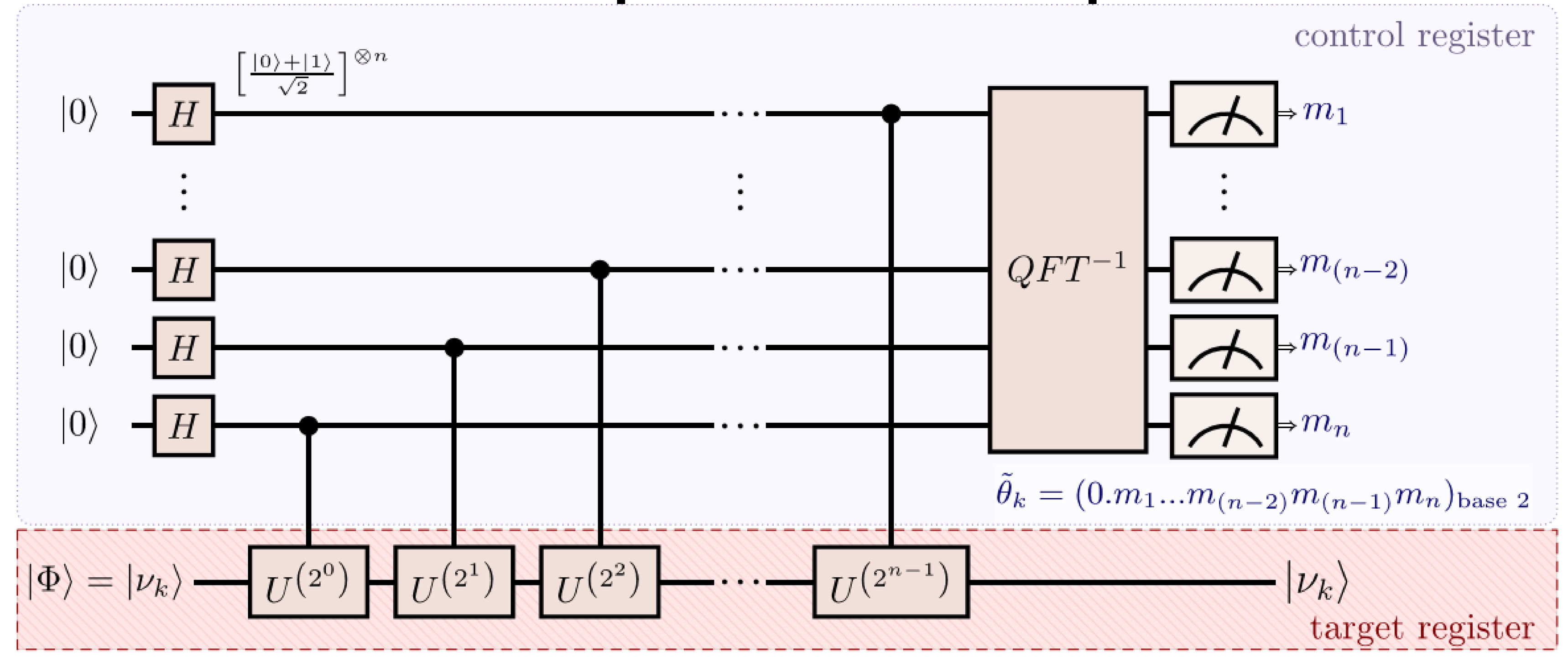
[3] S. Lloyd, M. Mohseni, and P. Rebentrost. "Quantum principal component analysis". *Nature Physics* 10.9 (2014), pp. 631–633

[4] T. Byrnes, G. Forster, and L. Tessler. "Generalized Grover's Algorithm for Multiple Phase Inversion States". *Phys. Rev. Lett.* 120 (2018), p. 060501. doi: 10.1103/PhysRevLett.120.060501

[5] A. Daskin and S. Kais. "Direct application of the phase estimation algorithm to find the eigenvalues of the Hamiltonians". *Chemical Physics* 514 (2018), pp. 87–94

[6] A. Aspuru-Guzik, A. D. Dutoi, P. J. Love, and M. Head-Gordon. "Simulated quantum computation of molecular energies". *Science* 309.5741 (2005),pp. 1704–1707


### Traditional implementation of qubit-PEA



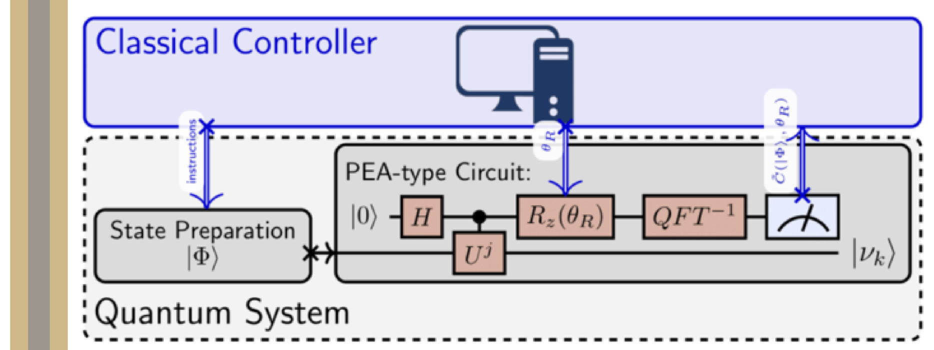

$n$ qubits all initialized to $|0\rangle$ make up the control register. If the unitary of interest $U$ is $2^b$-by- $2^b$ dimensional then $b$ qubits comprise the target register which can be initialized to any state $|\Phi\rangle$. In the diagrammed circuit, $|\Phi\rangle$ is initialized as eigenstate $|\nu_k\rangle$. Each control qubit controls whether a quantum gate will be applied to the target qubits; the gate is applied when the control is in $|1\rangle$ and not applied when the control is $|0\rangle$. $H$ represents the Hadamard gate, which converts $|0\rangle$ to $(|0\rangle + |1\rangle)/\sqrt{2}$. $QFT^{-1}$ is the inverse of the $2^n$-dimensional quantum Fourier transform gate. Upon measurement, each control qubit collapses to either 0 or 1. The eigenstate $|\nu_k\rangle$ corresponds to eigenvalue $e^{i2\pi\theta_k}$ and eigenphase $\theta_k$. If $\theta_k \in [0,1)$ can be represented with a base-2 number of $n$ or fewer digits, the measured values of the control qubits $\{m_x\}$ will represent this value:

$$\theta_k = (0.m_1 m_2 \dots m_{n-1} m_n)_{\text{base } 2}.$$

Otherwise, the measured values will represent an approximation $\tilde{\theta_k}$ with precision $\pm 2^n$ with high probability. The solid horizontal lines represent qubits in the circuit. Double lines represent classical information, typically generated by a measurement gate. The purple region outlines the n control qubits; the red region outlines the b target qubits.

## Novel Statistical PEA (SPEA)(Moore, 2021)

### SPEA Schematic

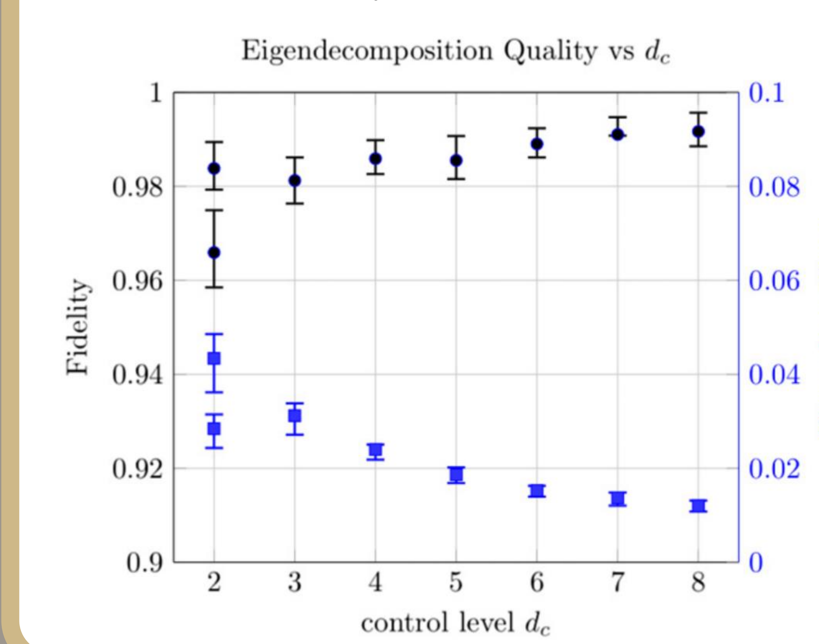



Expanding on the work from (Lu, 2020), and integrating ideas from the iterative PEA (IPEA), the statistical PEA (SPEA) was proposed and studied. The SPEA utilizes a variational scheme: a classical controller controls the input state and the rotation gate of an IPEA circuit. The statistics of the measured state serve as an objective function: the prepared state and applied rotation are the eigen-state and –phase when the control qudit deterministically is measured in bin $|0\rangle$.

### Simulated Results

The results from the quantum portion of the SPEA were simulated on both a classical computer and on IBM's Qiskit platform. The classical controller was implemented using a modified gradient search algorithm. The SPEA successfully arrives at valid eigenstate-phase pairs regardless starting conditions, as shown in the table (right). The SPEA was tested on two 3-dimensional unitaries and one 4-dimensional unitary.

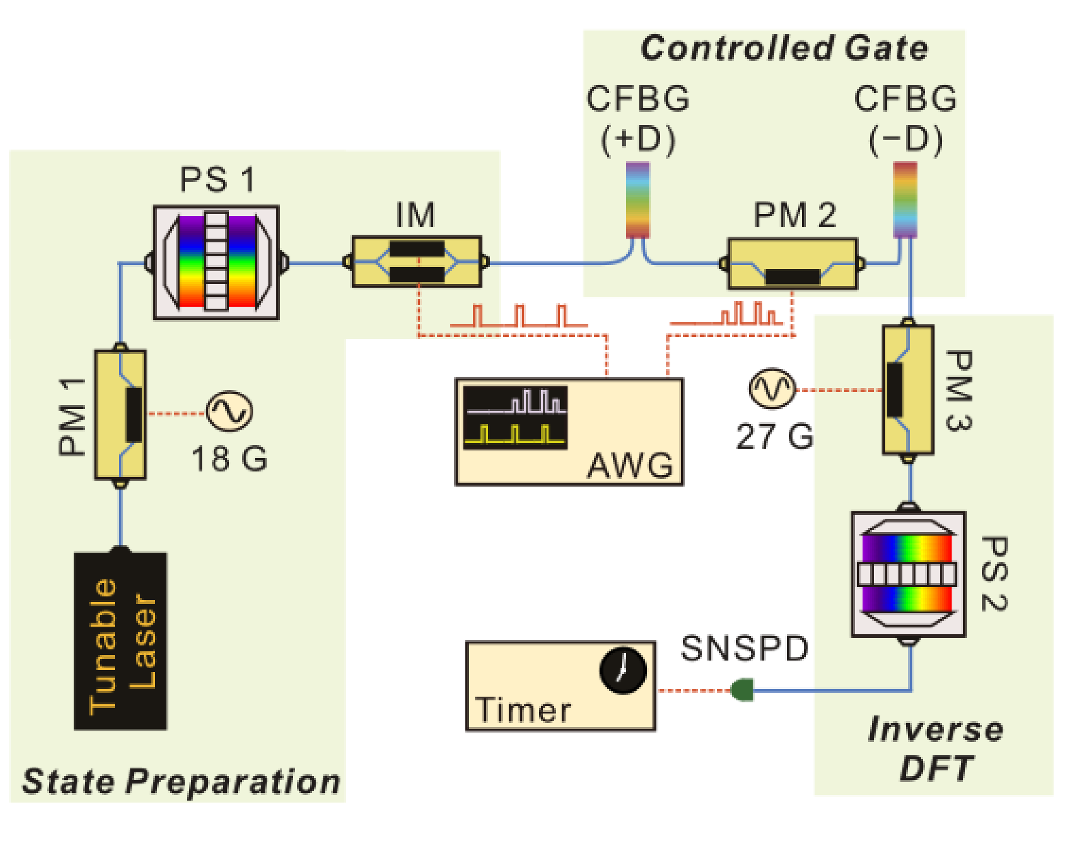
| Unitary | Input state | Abs. inner product | Iteration Mean | Iteration S.D. | Mean phase error |
|---|---|---|---|---|---|
| $U_1$ | (0.1951, 0.9808) | 0.98 | 6.20 | 2.82 | $1.099 \times 10^{-2}$ |
| | (0.3827,0.9239) | 0.92 | 8.15 | 3.41 | $1.005 \times 10^{-2}$ |
| | (0.7071, 0.7071) | 0.71 | 8.90 | 3.34 | $1.005 \times 10^{-2}$ |
| $U_2$ | (0, 0, 0.7432, 0.6690) | 0.99 | 5.85 | 8.14 | $2.083 \times 10^{-2}$ |
| | (0, 0, 0.6690, 0.7432) | 0.99 | 6.7 | 10.42 | $2.168 \times 10^{-2}$ |
| | (0, 0, 1, 0) | 0.71 | 17.7 | 6.06 | $1.663 \times 10^{-2}$ |
| | (1, 0, 0, 0) | 0.71 | 23.05 | 11.22 | $2.167 \times 10^{-2}$ |
| | (0.7071, 0, 0.7071, 0) | 0.50 | 21.3 | 10.71 | $2.262 \times 10^{-2}$ |
| $U_3$ | (−0.1379, 0, 0, 0.9904) | 0.99 | 1.15 | 0.36 | $1.885 \times 10^{-2}$ |
| | (0, 0.7807, 0.6247, 0) | 0.99 | 1.1 | 0.3 | $1.508 \times 10^{-2}$ |
| | (0, 1, 0, 0) | 0.71 | 4.35 | 4.17 | $1.414 \times 10^{-2}$ |
| | (0.7071, 0, 0, 0.7071) | 0.62 | 4.15 | 1.01 | $1.570 \times 10^{-2}$ |
| | (0.5774, 0.5774, 0, 0.5774) | 0.51 | 21.5 | 11.06 | $2.199 \times 10^{-2}$ |




The SPEA can be iterated to perform a full spectral decomposition, recovering all eigenstate-phase pairs. This functionality was tested on the (quantum chemistry relevant) 16-by-16 representation of the water molecule. The quality of the decomposition (both in overall matrix fidelity and minimum average eigenphase error) improved as higher-dimensional control circuits were used. This illustrates not only the strength of the SPEA itself, but also indicates some unique advantages may be offered by high-dimensional algorithms.

## Photonic Realization of *high-dimensional* PEA for diagonal unitary systems (Lu, 2020)

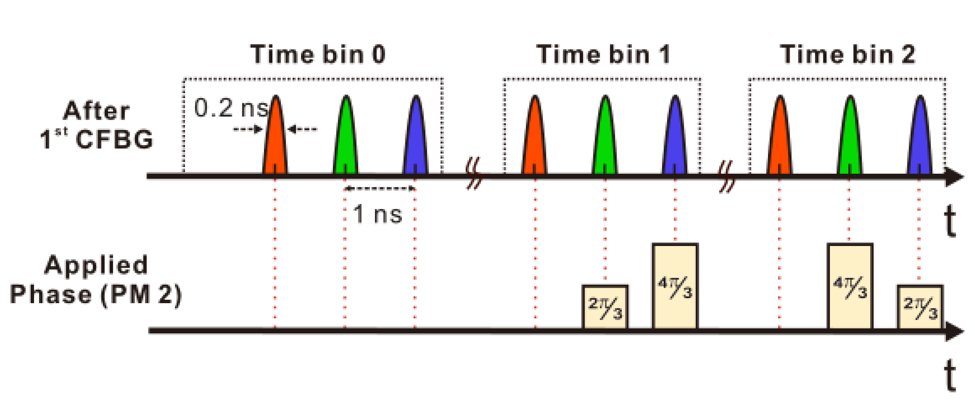
### Experimental setup: frequency-controlled time-target PEA



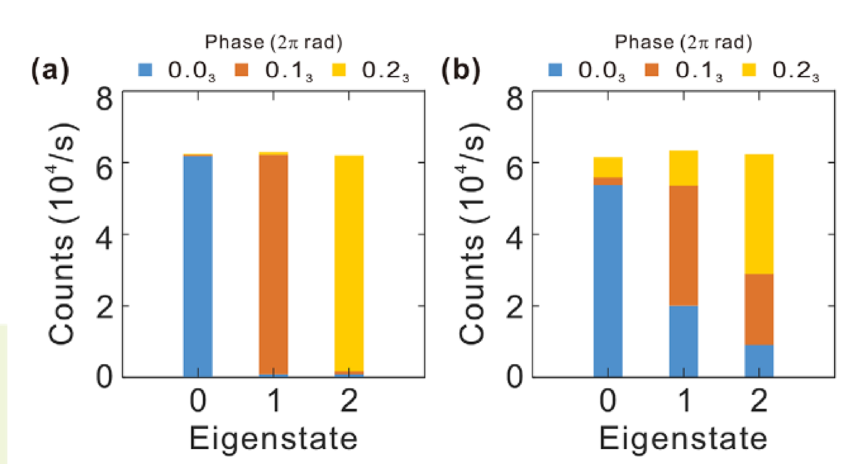

**PM/IM**: Electro-optic phase/intensity modulator
**PS**: Fourier-transform pulse shaper
**CFBG**: Chirped fiber Bragg grating
**SNSPD**: Superconducting nanowire single-photon detector
**AWG**: Arbitrary waveform generator.
Both radio-frequency oscillators (18 and27 GHz) are synchronized to the 10 MHz reference clock of the AWG.

A CW laser is carved into 3 frequency bins, spaced at 54GHz. Three narrow (~.2ns) spaced by 6ns are carved. The CFBG separates the frequencies within each time bin, allowing the PM to apply a different phase to each bin. This realizes a time-controlled (diagonal) unitary gate, illustrated in the lower figure. The next CFBG closes the time bin. The final PM and PS apply a probabilistic inverse (discrete) Fourier transform operation on the frequency degree of freedom.

### Description of controlled-phase gate




### Results



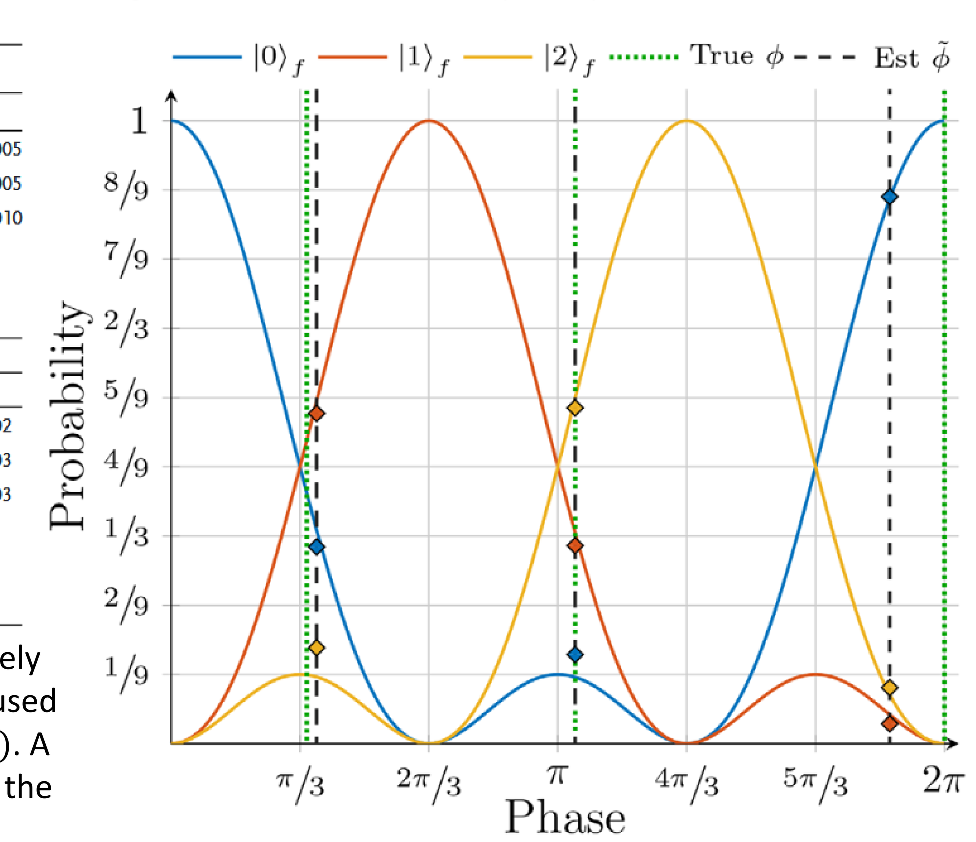

Each single photon counted is equivalent to one run of the traditional PEA circuit. As both the control and target degrees of freedom are realized on a single photon, a CW laser source can be used, effectively running the PEA many times. On the left, figure (a) shows the results for a diagonal unitary with eigenphases $(0, 2\pi/3, 4\pi/3)$. Data (b) shows the results for a diagonal unitary with eigenphases $(0,\ .3511\pi,\ 1.045\pi)$. The implementation is considered successful, as the data allows each eigenphase to be determined to the nearest trit ($0\ 2\pi/3$, or $4\pi/3$). As we have special knowledge that the input states are eigenstates, additional analysis is possible. See below.

### Novel statistical analysis

| $\hat{U}_1$ | | | |
|---|---|---|---|
| **Eigenstate** | $\|0\rangle_t$ | $\|1\rangle_t$ | $\|2\rangle_t$ |
| $E_0$ | $.9948 \pm .0004$ | $.0101 \pm .0004$ | $.0122 \pm .0005$ |
| $E_1$ | $.0023 \pm .0002$ | $.9805 \pm .0009$ | $.0120 \pm .0005$ |
| $E_2$ | $.0029 \pm .0002$ | $.0094 \pm .0004$ | $.9758 \pm .0010$ |
| True Phase, $\phi$ | 0 | $2\pi/3$ | $4\pi/3$ |
| Est. Phase, $\tilde{\phi}$ | $1.972\pi$ | $.612\pi$ | $1.394\pi$ |
| Error, $\frac{\|\phi-\tilde{\phi}\|}{2\pi}$ | 1.4% | 2.7% | 3.0% |

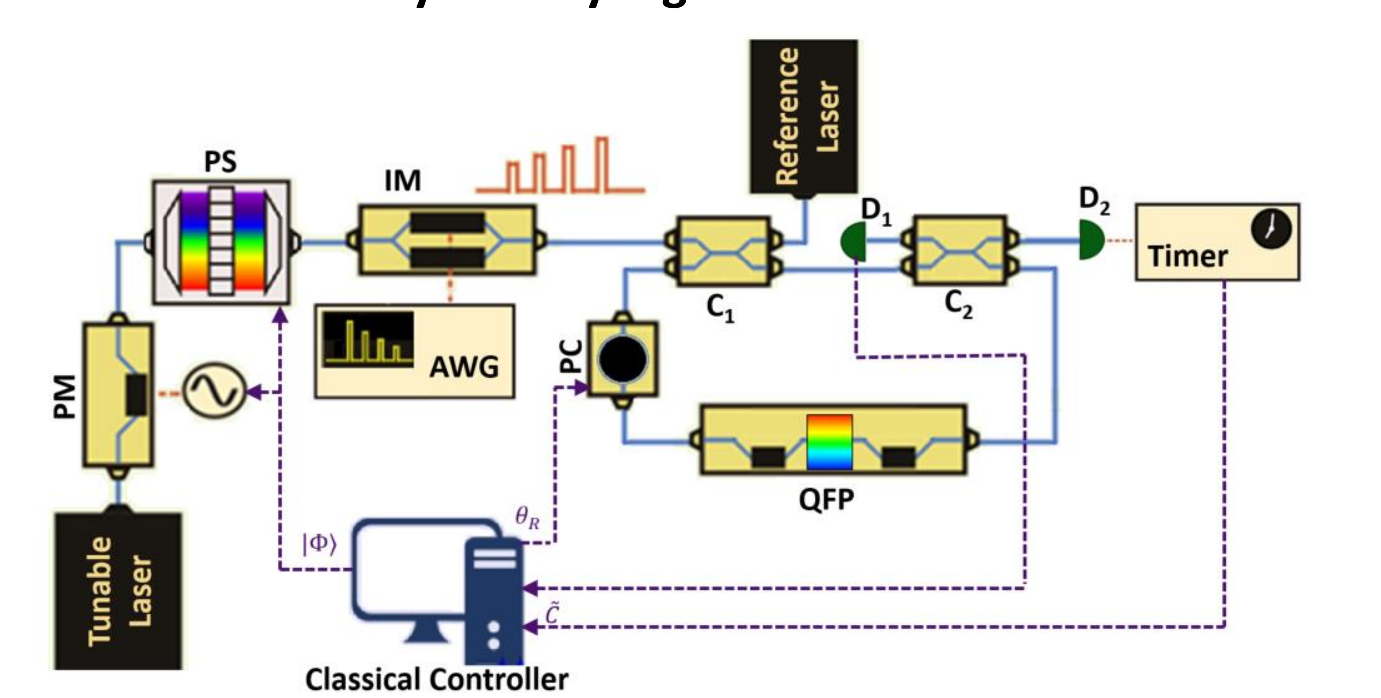
| $\hat{U}_2$ | | | |
|---|---|---|---|
| **Eigenstate** | $\|0\rangle_t$ | $\|1\rangle_t$ | $\|2\rangle_t$ |
| $E_0$ | $.878 \pm .002$ | $.316 \pm .003$ | $.143 \pm .002$ |
| $E_1$ | $.032 \pm .001$ | $.530 \pm .003$ | $.318 \pm .003$ |
| $E_2$ | $.090 \pm .002$ | $.154 \pm .002$ | $.539 \pm .003$ |
| True Phase, $\phi$ | 0 | $.3511\pi$ | $1.045\pi$ |
| Est. Phase, $\tilde{\phi}$ | $1.859\pi$ | $.377\pi$ | $1.045\pi$ |
| Error, $\frac{\|\phi-\tilde{\phi}\|}{2\pi}$ | 7.1% | 1.3% | 0.0% |

As many trials of the photon PEA have effectively been run, the statistics of the outputs can be used to determine a phase anywhere on $\theta \in [0, 2\pi)$. A visual representation of this fitting is made on the right. Results and error tabulated above.




## Future Work

### Arbitrary unitary high-dimensional SPEA




**PM**: Electro-optic phase/intensity modulator; **PS**: pulse shaper; **IM**: intensity modulator; **QFP**: Quantum frequency processor; **C**: 2-by-2 optical couplers; **PC**: phase controller (i.e. fiber shifter); **AWG**: Arbitrary waveform generator

A proposed setup for realizing a high-dimensional time-controlled frequency-target SPEA. A state preparation setup like (Lu, 2020) carves the CW laser into three frequency bins. The IM then carves the signal into four time bins prior to entering a loop. A QFP placed in a loop implements an arbitrary unitary operation on the frequency bins. The first (second, third) time-bin to enter the loop will make three (two, one) round trips prior to the last time bin entering the loop. This realizes a time-controlled unitary operation. The overlapping of the time bins realizes a pseudo-inverse Fourier transform which allows the output in time bin $|0\rangle$ to be measured, satisfying the needs of the SPEA.


(Lu, 2020) and (Moore, 2021) are collaborative work between Weiner and Kais Labs at Purdue University.
Weiner Group: https://engineering.purdue.edu/~fsoptics
Kais Group: https://www.chem.purdue.edu/kais/

Work supported by National Science Foundation (NSF) under Award No. 1839191-ECCS and under Award No. 2124511-CHE


## Reported Works


(Lu, 2020) Hsuan-Hao Lu, Zixuan Hu, Mohammed Saleh Alshaykh, Alexandria Moore, Yuchen Wang, Poolad Imany, Andrew Weiner, and Sabre Kais. "Quantum phase estimation with time-frequency qudits in a single photon". *Advanced Quantum Technologies* 3.2 (2020), p. 1900074.

(Moore, 2021) Alexandria J Moore, Yuchen Wang, Zixuan Hu, Sabre Kais, and Andrew M Weiner. "Statistical approach to quantum phase estimation". *New Journal of Physics* 23.11 (Nov. 2021), p. 113027. doi: 10.1088/1367-2630/ac320d