Note: Unless otherwise stated, you need to justify your answers to get the full credit.

Problem 1. (40 points) Consider the LTI system $\dot{x} = Ax + Bu$, $y = Cx$, where

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 0 & 1 \\ -2 & -1 & -2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & -2 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} -2 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ -1 & 0 & -1 \end{bmatrix}}_{T^{-1}}, \quad B = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

(a) **(4 pts)** Compute e^{At} for $t \geq 0$.

(b) **(4 pts)** For the autonomous system $\dot{x} = Ax$, find its three modes and determine its stability.

(c) **(4 pts)** Find the set of all possible initial states $x(0) \in \mathbb{R}^3$ starting from which the solutions to $\dot{x} = Ax$ satisfy $x(t) \to 0$ as $t \to \infty$. If instead it is desired that $x(t)$ remains bounded for all $t \geq 0$, what is the set of all possible $x(0)$?

(d) **(4 pts)** Is the given LTI system controllable? Find the reachable subspace (controllable subspace) of the system.

(e) **(4 pts)** Is the given LTI system observable? What is its unobservable subspace?

(f) **(4 pts)** Using the state feedback control $u = -Kx$, can you find a proper gain matrix $K \in \mathbb{R}^{1\times 3}$ so that the resulting closed-loop system $\dot{x} = A_{\text{cl}}\, x$ is stable?

(g) **(4 pts)** Using the state feedback control $u = -Kx$, can you find a proper gain matrix $K \in \mathbb{R}^{1\times 3}$ so that the resulting closed-loop system $\dot{x} = A_{\text{cl}}\, x$, $y = Cx$, satisfies $y(t) \to 0$ as $t \to \infty$ for all $x(0) \in \mathbb{R}^3$? Find such a K if the answer is yes; otherwise, state your reasons.

(h) **(4 pts)** Can you find $L \in \mathbb{R}^3$ such that $A - LC$ is stable? More generally, what are the possible sets of eigenvalues of $A - LC$ for arbitrary choices of L.

(i) **(4 pts)** Find the transfer function $H(s) = \frac{Y(s)}{U(s)}$. Is the system BIBO stable, i.e., for $x(0) = 0$ and bounded input $u(t)$, the system output $y(t)$ will remain bounded for all $t \geq 0$?

(j) **(4 pts)** Suppose $x(0) = \begin{bmatrix} -2 & -4 & 4 \end{bmatrix}^T$ and $u(t) = 1$ for all $t \geq 0$. Find $y(t)$ for $t \geq 0$ for the given LTI system.

Problem 2. (20 points) To the extent possible, find the fundamental matrix $\Phi(t)$ of the following LTV system

$$\dot{x}(t) = \begin{bmatrix} -t & 1 \\ 0 & -1 \end{bmatrix} x(t).$$

Problem 3. (20 points) A discrete-time LTI system is given as

$$x[k+1] = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k], \quad k = 0, 1, \ldots.$$

From the initial state $x[0] = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, find the control input $u[0]$, $u[1]$, and $u_2[0]$ that can steer the state to zero at time $k = 3$ (i.e., $x[3] = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$) with the least control energy $|u[0]|^2 + |u[1]|^2 + |u[2]|^2$.

Problem 4. (20 points) Find all the equilibrium points of the following nonlinear system and determine the local stability around each of them, if possible:

$$\dot{x}_1 = (x_1^2 - 1)(x_2 - 2)$$
$$\dot{x}_2 = -x_2(x_1^2 + 1).$$

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