August 2014 QE

Continuous-Time Fourier Transform: $X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Continuous-Time Fourier Transform Pair: $\mathcal{F}\left\{\frac{\sin(Wt)}{\pi t}\right\} = rect\left\{\frac{\omega}{2W}\right\}$ where $rect(x) = 1$ for $|x| < 0.5$ and $rect(x) = 0$ for $|x| > 0.5$.

Continuous-Time Fourier Transform Property: $\mathcal{F}\{x_1(t)x_2(t)\} = \frac{1}{2\pi}X_1(\omega) * X_2(\omega)$, where $*$ denotes convolution, and $\mathcal{F}\{x_i(t)\} = X_i(\omega)$, $i = 1, 2$.

Discrete-Time Fourier Transform: $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Problem 1. [50 pts]

Problem 1 (a). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. That is, the Fourier Transform of the analog signal $x_a(t)$ is exactly zero for $|\omega| > 20$ rads/sec. This signal is sampled at a rate $\omega_s = 60$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{60}$ sec. This yields the discrete-time sequence

$$x[n] = x_a(nT_s) = \left\{\frac{60}{2\pi}\right\}\left\{\frac{\sin(\frac{\pi}{3}n)}{\pi n} + \frac{\sin(\frac{2\pi}{3}n)}{\pi n}\right\} \qquad \text{where: } T_s = \frac{2\pi}{60}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where: } T_s = \frac{2\pi}{60} \text{ and } h(t) = T_s\frac{\pi}{10}\frac{\sin(10t)}{\pi t}\frac{\sin(30t)}{\pi t}$$

Problem 1 (b). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 60$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where: } T_s = \frac{2\pi}{60} \text{ and } h(t) = T_s\frac{\pi}{15}\frac{\sin(15t)}{\pi t}\frac{\sin(25t)}{\pi t}$$

Problem 1 (c). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 60$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where: } T_s = \frac{2\pi}{60} \text{ and } h(t) = T_s\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(35t)}{\pi t}$$

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Problem 1 (d). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. That is, the Fourier Transform of the analog signal $x_a(t)$ is exactly zero for $|\omega| > 20$ rads/sec. This signal is sampled at a rate $\omega_s = 50$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{50}$ sec. This yields the discrete-time sequence

$$x[n] = x_a(nT_s) = \left\{\frac{50}{2\pi}\right\}\left\{\frac{\sin(\frac{2\pi}{5}n)}{\pi n} + \frac{\sin(\frac{4\pi}{5}n)}{\pi n}\right\} \qquad \text{where: } T_s = \frac{2\pi}{50}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where: } T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s\frac{\sin(20t)}{\pi t}$$

Problem 1 (e). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 50$ rads/sec., but reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where: } T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s\frac{\sin(35t)}{\pi t}$$

Problem 1 (f). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 60$ rads/sec., but reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where: } T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s\frac{\sin(15t)}{\pi t}$$

Problem 1 (g). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at a rate $\omega_s = 30$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{30}$ sec, yielding the following discrete-time sequence:

$$x[n] = x_a(nT_s) = \left\{\frac{30}{2\pi}\right\}\left\{\frac{\sin(\frac{2\pi}{3}n)}{\pi n} + \frac{\sin(\frac{4\pi}{3}n)}{\pi n}\right\} \qquad \text{where: } T_s = \frac{2\pi}{30}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a closed-form expression for the reconstructed signal $x_r(t)$. Show all work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where: } T_s = \frac{2\pi}{30} \quad \text{and} \quad h(t) = T_s\frac{\sin(15t)}{\pi t}$$

Problem 1 (h). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 30$ rads/sec., where $\omega_s = 2\pi/T_s$ and the time between samples is $T_s = \frac{2\pi}{30}$ sec, but at a different starting point. This yields the Discrete-Time $x[n]$ signal below:

$$x_\epsilon[n] = x_a(nT_s + 0.5T_s) = \left\{\frac{30}{2\pi}\right\}\left\{\frac{\sin(\frac{2\pi}{3}(n+0.5))}{\pi(n+0.5)} + \frac{\sin(\frac{4\pi}{3}(n+0.5))}{\pi(n+0.5)}\right\} \qquad \text{where: } T_s = \frac{2\pi}{30}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x_\epsilon[n]h\left(t - (n+0.5)T_s\right) \qquad \text{where: } T_s = \frac{2\pi}{30} \quad \text{and} \quad h(t) = T_s\frac{\sin(15t)}{\pi t}$$

Problem 2 starts on next page.

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Problem 2. [50 points] Consider a causal FIR filter of length $M = 8$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \frac{\sin\left[\frac{5\pi}{8}(n+\ell 8)\right]}{\pi(n+\ell 8)} \{u[n] - u[n-8]\}$$

(a) Determine all 8 numerical values of the 8-pt DFT of $h_p[n]$, denoted $H_8(k)$, for $0 \leq k \leq 7$. List the values clearly: $H_8(0) = ?$, $H_8(1) = ?$, $H_8(2) = ?$, $H_8(3) = ?$, $H_8(4) = ?$, $H_8(5) = ?$, $H_8(6) = ?$, $H_8(7) = ?$.

(b) Consider the sequence $x[n]$ of length $L = 8$ below, equal to a sum of 8 finite-length sinewaves.

$$x[n] = \sum_{k=0}^{7} e^{jk\frac{2\pi}{8}n} \{u[n] - u[n-8]\}$$

$y_8[n]$ is formed by computing $X_8(k)$ as an 8-pt DFT of $x[n]$, $H_8(k)$ as an 8-pt DFT of $h_p[n]$, and then $y_8[n]$ as the 8-pt inverse DFT of $Y_8(k) = X_8(k)H_8(k)$. Express the result $y_8[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.

Next, consider a causal FIR filter of length $M = 8$ with impulse response as defined below:

$$h_p[n] = \sum_{\ell=-\infty}^{\infty} \frac{8}{3} \frac{\sin\left[\frac{5\pi}{8}(n+\ell 8)\right]}{\pi(n+\ell 8)} \frac{\sin\left[\frac{3\pi}{8}(n+\ell 8)\right]}{\pi(n+\ell 8)} \{u[n] - u[n-8]\}$$

(c) Determine all 8 numerical values of the 8-pt DFT of $h_p[n]$, denoted $H_8(k)$, for $0 \leq k \leq 7$. List the values clearly: $H_8(0) = ?$, $H_8(1) = ?$, $H_8(2) = ?$, $H_8(3) = ?$, $H_8(4) = ?$, $H_8(5) = ?$, $H_8(6) = ?$, $H_8(7) = ?$.

(d) Consider the sequence $x[n]$ of length $L = 8$ below, equal to a sum of 8 finite-length sinewaves.

$$x[n] = \sum_{k=0}^{7} e^{jk\frac{2\pi}{8}n} \{u[n] - u[n-8]\}$$

$y_8[n]$ is formed by computing $X_8(k)$ as an 8-pt DFT of $x[n]$, $H_8(k)$ as an 8-pt DFT of $h_p[n]$, and then $y_8[n]$ as the 8-pt inverse DFT of $Y_8(k) = X_8(k)H_8(k)$. Express the result $y_8[n]$ as a weighted sum of finite-length sinewaves similar to how $x[n]$ is written above.