AC-3
August 2014 QE

1. (20 pts) Find the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

that comes as close as possible to the three data points:

$$(x_1, y_1) = (1, 0), \quad (x_2, y_2) = (0, \sqrt{2}), \quad (x_3, y_3) = (1, 1).$$

2. (20 pts) Use the simplex method to solve the following linear program,

$$\begin{array}{ll} \text{maximize} & x_1 + 2x_2 \\ \text{subject to} & -2x_1 + x_2 \le 2 \\ & x_1 - x_2 \ge -3 \\ & x_1 \le 3 \\ & x_1 \ge 0, \; x_2 \ge 0. \end{array}$$

3. (20 pts) Consider the following model of a linear, discrete, time-invariant system,

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k, \quad 0 \le k \le N-1,$$

with a specified initial condition $\boldsymbol{x}_0$ and a specified final state $\boldsymbol{x}_N = \boldsymbol{x}_f$, where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times m}$, and $N \ge n$. We assume that the pair $(\boldsymbol{A}, \boldsymbol{B})$ is reachable. Use the Lagrange multiplier approach to calculate the optimal control sequence

$$\{\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N-1}\}$$

that transfers $\boldsymbol{x}_0$ to $\boldsymbol{x}_f$ while minimizing the quadratic performance index

$$J_N = \frac{1}{2}\sum_{k=0}^{N-1} \boldsymbol{u}_k^\top \boldsymbol{R}\boldsymbol{u}_k,$$

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where $\boldsymbol{R} = \boldsymbol{R}^\top > 0$.

Hint: Define the composite input vector

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_0^\top & \boldsymbol{u}_1^\top & \cdots & \boldsymbol{u}_{N-1}^\top \end{bmatrix}^\top$$

and the symmetric block-diagonal positive-definite matrix

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R} & \cdots & \boldsymbol{O} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{R} \end{bmatrix}.$$

It is then easy to verify that the performance index J_N can be represented as

$$J_N = \frac{1}{2}\boldsymbol{u}^\top \boldsymbol{L}\boldsymbol{u}.$$

Next, write the plant model in the form

$$\boldsymbol{M}\boldsymbol{u} = \boldsymbol{f}$$

for some matrix $\boldsymbol{M}$ and a vector $\boldsymbol{f}$.

(i) **(5 pts)** Give expressions for $\boldsymbol{M}$ and $\boldsymbol{f}$. Note that the expression for $\boldsymbol{f}$ is in terms of $\boldsymbol{x}_f$ and $\boldsymbol{x}_0$.

(ii) **(5 pts)** Represent the problem of optimal transfer of the system from the initial state $\boldsymbol{x}_0$ to the final state $\boldsymbol{x}_f$ as a constrained optimization problem.

(iii) **(10 pts)** Obtain a closed-form expression for $\boldsymbol{u}$.

4. (20 pts) Consider a square matrix $\boldsymbol{Q}$ partitioned as follows:

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{bmatrix},$$

where $\boldsymbol{Q}_{11}$ and $\boldsymbol{Q}_{22}$ are square submatrices. If $\boldsymbol{Q}_{11}$ is nonsingular, then we can write

$$\begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Delta} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12} \\ \boldsymbol{O} & \boldsymbol{I} \end{bmatrix},$$

where $\Delta = Q_{22} - Q_{21}Q_{11}^{-1}Q_{12}$ is called the *Schur complement* of Q_{11}. Suppose now that Q is symmetric, that is, $Q = Q^{\top}$.

(i) **(10 pts.)** Formulate necessary and sufficient conditions for Q to be positive definite in terms of Q_{11}, Q_{12}, and Q_{22};

(ii) **(10 pts.)** Assume that Q_{22} is nonsingular. Find an expression for the Schur complement of Q_{22}.

5. (20 pts) Given a monotone non-decreasing function g of single variable, that is, $g(r_1) \leq g(r_2)$ for $r_1 < r_2$. The function g is also convex. Let f be a convex function on a convex set $\Omega \subseteq \mathbb{R}^n$. Show that the composite function $g(f(\boldsymbol{x}))$ is convex on Ω.

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