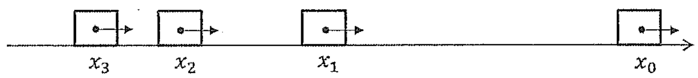
AC-2
August 2014 QE

LTI and LT Systems – State-Space Approach
August 2014



Problem 1. (40 points) Consider four cars moving in a platoon from left to right on the highway, whose initial positions at time $t = 0$ are given by $x_0(0) = 10$, $x_1(0) = 4$, $x_2(0) = 1$, $x_3(0) = 0$.

(a) **(20 pts)** Assume the locations of both the leading car (denoted by the index "0") and the trailing car (denoted by the index "3") are fixed: $x_0(t) \equiv 10$, $x_3(t) \equiv 0$, for all $t \geq 0$; while the two cars in the middle (car 1 and car 2) follow the dynamics:

$$\dot{x}_1(t) = \frac{x_0(t) + x_2(t)}{2} - x_1(t), \quad \dot{x}_2(t) = \frac{x_1(t) + x_3(t)}{2} - x_2(t), \quad t \geq 0, \tag{1}$$

i.e., each of them moves instantaneously towards the average location of the two cars immediately before and after it.

(i) **(5 pts)** Reformulate the above dynamics as a 2-dimensional LTI state-space model (A, B) with state vector $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$ driven by a constant input vector.

(ii) **(5 pts)** Compute e^{At} using a suitable method.

(iii) **(10 pts)** Determine whether or not the state vector $x(t)$ converges as $t \to \infty$. If so, determine the limiting value $x(\infty)$.

For the next three subproblems, we will consider a scenario slightly different from part (a). Assume that only the position of the trailing car 3 is fixed: $x_3(t) \equiv 0$, $t \geq 0$. The leading car 0's position is now controlled by an external input $u(t)$ as:

$$\dot{x}_0(t) = u(t), \quad t \geq 0.$$

The positions of cars 1 and 2 in the middle still follow the dynamics in (1). The four cars have the same initial positions at time $t = 0$ as in part (a).

(b) **(10 pts)**

(i) **(4 pts)** With the new state vector $\tilde{x}(t) = \begin{bmatrix} x_0(t) & x_1(t) & x_2(t) \end{bmatrix}^T$ and input $u(t)$, write the new LTI state dynamics equation $(\tilde{A}, \tilde{B})$ for some proper matrices $\tilde{A}$ and $\tilde{B}$.

(ii) **(6 pts)** Suppose the goal is such that at time $t = 100$ the cars should be located at $x_0(100) = x_1(100) = 10$, $x_2(100) = 0$. Does there exist an input $u(t)$, $0 \leq t \leq 100$, to achieve the above goal? Justify your answer.

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(c) **(5 pts)** Suppose the input $u(t)$ can only be nonzero for $0 \le t \le 1$ and $u(t) \equiv 0$ for $t > 1$. Does there exist such a $u(t)$ to achieve the same goal in part (b), namely, $x_0(100) = x_1(100) = 10$, $x_2(100) = 0$? Justify your answer.

(d) **(5 pts)** Suppose the input $u(t)$ can only be nonzero for $0 \le t \le 1$, and $u(t) \equiv 0$ for $t > 1$. Does there exist such a $u(t)$ so that in steady state, $\lim_{t\to\infty} x_0(t) = \lim_{t\to\infty} x_1(t) = 10$ and $\lim_{t\to\infty} x_2(t) = 0$? Justify your answer.

Problem 2. (20 points) Consider the following LTI system:

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 5 & -4 & 2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} x.$$

Determine if the system satisfies each of the following five properties. You do **not** need to justify your answers. For each property, you will get **4 points** for the correct answer, **-1 point** for the wrong answer, and **0 point** for no answer.

controllable | observable | detectable | stabilizable | BIBO stable.

Problem 3. (20 points) Find the state transition matrix $\Phi(t, \tau)$ for the following linear time-varying system:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -t & 0 \\ -\cos t & -t \end{bmatrix}}_{A(t)} x(t), \quad t \ge 0.$$

Problem 4. (20 points) For the following nonlinear system, find all its equilibrium points and determine their local stability, if possible:

$$\dot{x}_1 = x_1 x_2 - 2x_1$$
$$\dot{x}_2 = x_1 - x_2 - 1$$

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