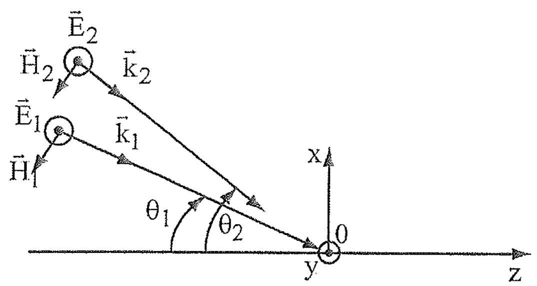
Two electromagnetic plane waves propagate through vacuum in directions defined by $\vec{k}_1$ and $\vec{k}_2$ as shown, where $\vec{k}_i = \frac{2\pi}{\lambda}\left(-\sin\theta_i\hat{x} + \cos\theta_i\hat{z}\right)$ for i = 1, 2. For each, the electric wave can be expressed as $\vec{E}_i = \vec{E}_{i,o}e^{-j\left(\vec{k}_i\cdot\vec{r}-\omega_i t\right)}$.



A) **(26 points)** For $\omega_1 = \omega_2$, determine the distance Λ between the time-averaged power density maxima in the plane defined by z = 0. Express your result in terms of $\lambda\left(=2\pi/\left|\vec{k}\right|\right)$, θ_1 and θ_2. Determine Λ for $\lambda = 1.0\ \mu\text{m}, \theta_1 = 0.01$ rad and $\theta_2 = -0.01$ rad.

B) **(27 points)** Now let ω_1 and ω_2 be similar to, but slightly different from one another (i.e., $\left|\omega_1 - \omega_2\right| \ll \omega_1, \omega_2$), and let $\theta_1 = -\theta_2$. Determine the velocity v of the time-averaged power density peaks in the z = 0 plane in terms of $\lambda, \theta_1, \theta_2$, and $\Delta\omega = \omega_1 - \omega_2$. Estimate v for $\lambda = 1.0\ \mu\text{m}, \theta_1 = -\theta_2 = 0.01$ rad and $\Delta\omega = 2\pi\times10^7$ rad/sec.

C) **(27 points)** Consider the two waves of frequency ω_1 and ω_2 (again with $\left|\omega_1 - \omega_2\right| \ll \omega_1, \omega_2$) as they propagate collinearly $\left(\theta_1 = \theta_2 = 0\right)$ through a dispersive, non-absorbing, isotropic, non-magnetic medium. The relative permittivity of the medium is given by

$\varepsilon_r(\omega) = \varepsilon_r\left(\omega_o\right) + \varepsilon_r'\left(\omega_o\right)\left(\omega - \omega_o\right) + \ldots$, where $\varepsilon_r'\left(\omega_o\right) = \left.\frac{d\varepsilon_r(\omega)}{d\omega}\right|_{\omega=\omega_o}$ and ω_o is the average frequency given by $\omega_o = \left(\omega_1 + \omega_2\right)/2$. Derive an expression for the velocity of the time-averaged power density peak of this waveform, travelling in the +z direction. Your answer should be in terms of $\omega_o, c\left(=(\varepsilon_o\mu_o)^{-1/2}\right), \varepsilon_r\left(\omega_o\right)$ and $\varepsilon_r'\left(\omega_o\right)$. Estimate this velocity for

$\lambda_o = 1.0\ \mu\text{m}$ (wavelength in free space), $\varepsilon_r\left(\omega_o\right) = 2, \varepsilon_r'\left(\omega_o\right) = \frac{10^{-11}}{2\pi}(\text{rad/sec})^{-1}$ and $\Delta\omega = 2\pi\times10^7$ rad/sec.

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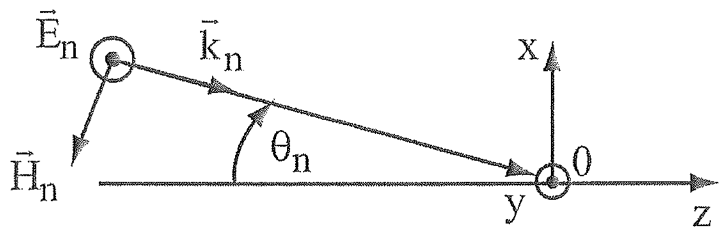
D) **(20 points)** Finally, we consider a large number N of plane waves, each propagating through vacuum (impedance $\eta \approx 377\ \Omega$) at an angle $\theta_n = n\Delta\theta$ with respect to the z axis, where n is an integer between $-N/2$ and $+N/2$. Let $N\Delta\theta \ll 1$.



$$E_n = E_o e^{-j(\vec{k}_n \cdot \vec{r} - \omega_o t)}$$

$$\vec{k}_n = \frac{2\pi}{\lambda}(-\sin\theta_n \hat{x} + \cos\theta_n \hat{z}).$$

The amplitude E_o and frequency ω_o are the same for all waves.

a) What is the peak time-averaged power density in the z = 0 plane in terms of N, E_o and η?

b) Show that the distance from the axis to the first zero of the time-averaged power density is $x_o = \dfrac{\lambda}{N\Delta\theta}$.

c) Find the distance x_m from the axis to the next time-averaged power density maximum.

d) What is the time-averaged power density of the peak at $x = x_m$, relative to that of the central peak?

[**Hint**: Represent the sum of these waves using a phasor diagram. The magnitude of each individual phasor is E_o and the phase difference between phasors is δ. Express δ as a function of x. As N gets large, the phasor sum representing the total field is the arc of a circle. Draw this phasor diagram for the field at the central peak. Repeat for the field at the first zero, and the next maximum.]

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Maxwell's Equations:

$$\nabla \cdot \mathbf{D} = \rho_v \qquad \oint\!\!\oint_S \mathbf{D}\cdot d\mathbf{S} = Q_{enc}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \oint\!\!\oint_S \mathbf{B}\cdot d\mathbf{S} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\oint\!\!\oint_S \mathbf{B}\cdot d\mathbf{S}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \oint_C \mathbf{H}\cdot d\mathbf{l} = I_{enc} + \frac{d}{dt}\oint\!\!\oint_S \mathbf{D}\cdot d\mathbf{S}$$

Poynting's Theorem:

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = -\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{H}) - \frac{\partial}{\partial t}(\mathbf{D}\cdot\mathbf{E}) - \mathbf{J}\cdot\mathbf{E}$$

Potentially Useful Vector Algebra

$$\nabla\cdot\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

Potentially Useful Integral Identities

$$\int \frac{dx}{x} = \ln x$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) \qquad \int \frac{x\,dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \qquad \int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln\left(x^2+a^2\right)$$

$$\int \frac{dx}{\left(x^2+a^2\right)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}} \qquad \int \frac{x\,dx}{\left(x^2+a^2\right)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}$$

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x \qquad \int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x$$

$$\int \sin^3 x\,dx = \frac{\cos^3 x}{3} - \cos x \qquad \int \cos^3 x\,dx = -\frac{\sin^3 x}{3} + \sin x$$

$$\int \sinh^2 x\,dx = \frac{1}{2}\left[-x + \sinh x\cosh x\right] \qquad \int \cosh^2 x\,dx = \frac{1}{2}\left[x + \sinh x\cosh x\right]$$

Other Information

$$v_g = \frac{d\omega}{dk},\quad k = \omega\sqrt{\mu\varepsilon}$$