CE-1

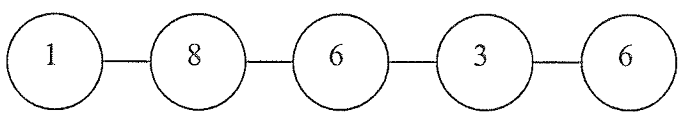
August 2011 QE

Q1 (20 points). Assume functions f and g such that *f(n)* is *O(gn))*. For each of the following statements, decide whether it is true or false and give a proof or counterexample.

(a) $2^{f(n)}$ is $O(2^{g(n)})$.

(b) $f(n)^2$ is $O(g(n)^2)$.

Q2 (30 points): Let G be a graph of n nodes connected in the form of a path with weights attached to its nodes. A subset of the nodes is called an *independent* set if no two of them are joined by an edge. Give an algorithm that takes this n-node path graph with weights and returns an independent set of maximum total weight (an example of a 5-node path graph with weights and with the resulting maximal weight of the independent set is shown below). The running time of your algorithm should be polynomial in n and independent of the values of the weights.



Note: The maximum weight of an independent set of this example is 14.

Q 3 (25 points): Given a graph G and a minimum spanning tree T, suppose that we decrease the weight of one of the edges in T. Show that T is still a minimum spanning tree for G. More formally, let T be a minimum spanning tree for G with edge weights given by weight function w. Choose one edge $(x, y) \in T$ and a positive number k, and define the weight function w' by

$$w'(u,v) = \begin{cases} w(u,v) & \textit{if } (u,v) \neq (x,y), \\ w(x,y) - k & \textit{if } (u,v) = (x,y). \end{cases}$$

Show that T is a minimum spanning tree for G with edge weights given by w'.

Q4 (25 points). Suppose you're helping to organize a summer sports camp, and the following problem comes up. The camp is supposed to have at least one counselor who's skilled at each of the n sports covered by the camp (baseball, volleyball, and so on). They have received job applications from m potential counselors. For each of the n sports, there is some subset of the m applicants qualified in the sport. The question is: For a given number $k < m$, is it possible to hire at most k of the counselors and have at least one counselor qualified in each of the n sports? We'll call this problem the *Efficient Recruiting Problem.*

Show that *Efficient Recruiting* is NP-complete.

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