FO-3
August 2007

Problem 1 (30 points):

Consider a circular waveguide as shown in Fig. 1. The waveguide wall is made of perfect electric conductor (PEC). It is uniformly filled with permittivity ε_0 and permeability μ_0.
(a) (10 points) Can this waveguide support TE_x (i.e. $E_x=0$) and TE_y (i.e. $E_y=0$) modes? Please refer to Fig. 1 for the coordinate system.
(b) (20 points) Justify your answer to question (a).

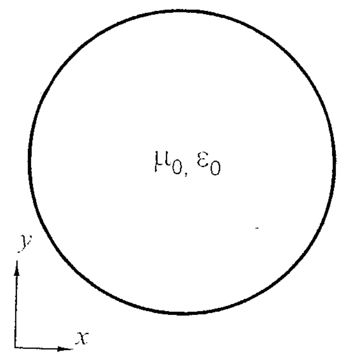

Fig. 1. A circular waveguide made of PEC.

Problem 2 (30 points):

Consider a rectangular waveguide loaded with two dielectric media as shown in Fig. 2. The permeability and permittivity of the two media are μ_0, ε_0, and μ_0, ε respectively. The waveguide wall is made of perfect electric conductor.

(a) (10 points) Can this waveguide support TM_z modes?
(b) (20 points) Justify your answer to question (a).

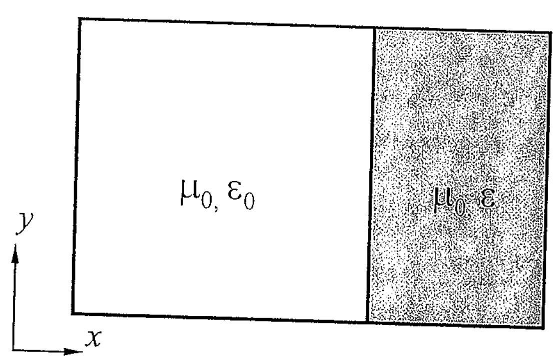

Fig. 2. A partially filled rectangular waveguide.

Problem 3 (40 points):

An infinitely long, uniform rectangular waveguide is filled with air having permittivity ε_0 and permeability μ_0 (see the figure below for its cross section). The waveguide wall is made of perfect magnetic conductor (PMC). Note that tangential magnetic field vanishes on a PMC surface. Derive the general expression of H_z for TE_z modes. (Hint: H_z satisfies $\nabla^2 H_z + k^2 H_z = 0,\ k^2 = \omega^2 \mu_0 \varepsilon_0$).

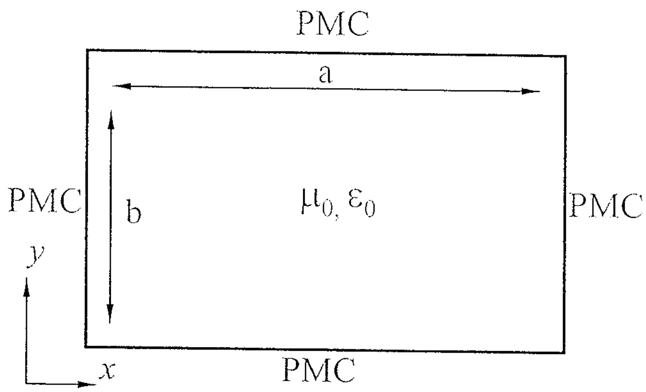

Fig. 3. A rectangular waveguide made of PMC.

Potentially Useful Formula:

Assuming $e^{j(\omega t - k_z z)}$ field dependence,

E_x, E_y, H_x, H_y can be written in terms of E_z and H_z as:

$$E_x = -\frac{1}{k^2 - k_z^2}(-jk_z \frac{\partial E_z}{\partial x} + j\omega\mu \frac{\partial H_z}{\partial y})$$

$$E_y = \frac{1}{k^2 - k_z^2}(jk_z \frac{\partial E_z}{\partial y} + j\omega\mu \frac{\partial H_z}{\partial x})$$

$$H_x = \frac{1}{k^2 - k_z^2}(j\omega\varepsilon \frac{\partial E_z}{\partial y} + jk_z \frac{\partial H_z}{\partial x})$$

$$H_y = -\frac{1}{k^2 - k_z^2}(j\omega\varepsilon \frac{\partial E_z}{\partial x} - jk_z \frac{\partial H_z}{\partial y})$$

Vector Potentials $\mathbf{A}$ and $\mathbf{F}$ satisfy:

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$

$$\nabla^2 \mathbf{F} + k^2 \mathbf{F} = -\varepsilon \mathbf{M}$$

Fields in terms of $\mathbf{A}$ and $\mathbf{F}$:

$$\mathbf{E} = -j\omega \mathbf{A} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla \bullet \mathbf{A} - \frac{1}{\varepsilon}\nabla \times \mathbf{F}$$

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} - j\omega \mathbf{F} - \frac{j}{\omega\mu\varepsilon}\nabla\nabla \bullet \mathbf{F}$$

Maxwell's Equations, in differential and integral form are: Other potentially useful relationships are:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \oint_c \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_s \vec{B} \cdot d\vec{s} \qquad \vec{D} = \varepsilon_o \vec{E} + \vec{P} = \varepsilon_o \vec{E}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \oint_c \vec{H} \cdot d\vec{\ell} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{s} \qquad \vec{B} = \mu_o \left(\vec{H} + \vec{M} \right) = \mu \vec{H}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \oint_s \vec{D} \cdot d\vec{s} = Q \qquad \lambda = \frac{2\pi}{\beta} = \frac{u}{f} = \frac{1}{f\sqrt{\mu\varepsilon}}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \oint_s \vec{B} \cdot d\vec{s} = 0 \qquad \varepsilon_0 = 8.854 \times 10^{-12}\, F/m$$

$$\mu_0 = 4\pi \times 10^{-7}\, H/m$$

Vector Differential Relationships in Cylindrical Coordinates (r, ϕ, z)

$$\vec{\nabla} V = \hat{a}_r \frac{\partial V}{\partial r} + \hat{a}_\phi \frac{\partial V}{r\partial \phi} + \hat{a}_z \frac{\partial V}{\partial z}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{\partial A_\phi}{r\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla} \times \vec{A} = \hat{a}_r \left(\frac{\partial A_z}{r\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{a}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \hat{a}_z \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\phi\right) - \frac{\partial A_r}{\partial \phi} \right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Divergence Thm: $\int_V \vec{\nabla} \cdot \vec{A} dv = \oint_S \vec{A} \cdot d\vec{s}$

Stokes Thm: $\int_V \vec{\nabla} \times \vec{A} \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{\ell}$

For lossless transmission lines:

$$Z_{in} = R_o \frac{Z_L + jR_o \tan\beta\ell}{R_o + jZ_L \tan\beta\ell}$$