1. (30 pts)

a) Starting from the differential forms of Maxwell's equations, derive a homogeneous Helmholtz equation (a wave equation) for the electric field valid in a linear, isotropic, source-free medium characterized by dielectric permittivity ε_0 and magnetic permeability μ_0. You may find this vector identity useful: $\nabla \times \nabla \times \vec{A} \equiv \nabla(\nabla \bullet \vec{A}) - \nabla^2 \vec{A}$, where $\vec{A}$ is an arbitrary 3-component vector.

b) Assuming $e^{j\omega t}$ time dependence, rewrite your electric field wave equation and give expressions for the electric field vector and the associated magnetic field vector valid for a time-harmonic, linearly polarized, plane wave solution to your wave equation (assume +z wave propagation).

c) Find an expression for the phase velocity of the wave in part b) in terms of ε_0 and μ_0. Is there dispersion in this medium?

2. (30 pts) A two-conductor transmission line is driven by a voltage source V_G (with internal resistance R_G) and is terminated by a load resistance R_L, as shown in Figure 1. The transmission line has a characteristic impedance of Z_0 and length L in the x direction. The voltage source output has the form $V_G = V_0 \cos(\omega t)$ volts.

a) Assume $R_L = Z_0/2$. What resistance Z_Q can you put in parallel with the line $\lambda/4$ in front of the load to eliminate reflections on the generator side of that resistance?

b) Assume $R_L = 0$ (short circuit termination), $Z_0 = R_G$ and $L = 2.75\lambda$. If the voltage source has been on for a long time, find an expression for the voltage between the conductors valid for all values of x.

c) What are the conditions on L and ω under which this transmission line system could be analyzed using static circuit theory techniques? Explain your answer briefly.

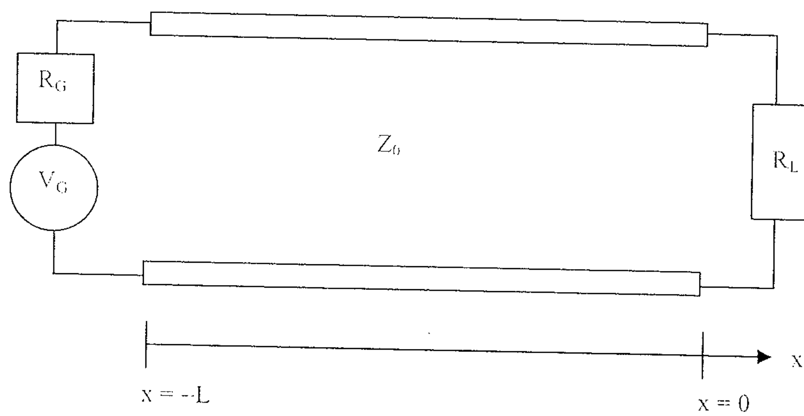

Figure 1: Transmission line (for problem 2).

3. (40 pts) A hollow waveguide with a rectangular cross section and perfectly conducting, metallic walls is shown in Figure 2. The relevant dimensions of the waveguide are also given. The hollow waveguide is filled with a perfect (lossless) dielectric characterized by ε_0 and μ_0. The waveguide is several hundred km long.

   a) Determine an expression for the cutoff frequency of this waveguide and evaluate your answer numerically.

   b) Write an expression for the waveguide propagation constant β in terms of the waveguide dimensions and the dielectric constants (ε_0 and μ_0).

   c) Write the instantaneous time domain expression for the z-component of the magnetic field (H_z) for the lowest-order mode that will propagate in this waveguide.

   d) A 12 GHz carrier signal is modulated by a 2.5 GHz data signal (5 GHz total modulation bandwidth) and is transmitted on this waveguide. In the time domain, the electric field for the modulated wave can be written as $E_0 \cos(\omega_1 t) \cos(\omega_2 t)$, where $\omega_1 = 5\pi x 10^9$ rad/s and $\omega_2 = 24\pi x 10^9$ rad/s. Determine what frequencies will exit the waveguide at the far end, and describe the waveguide's effects on the combination of the carrier and modulation waves.

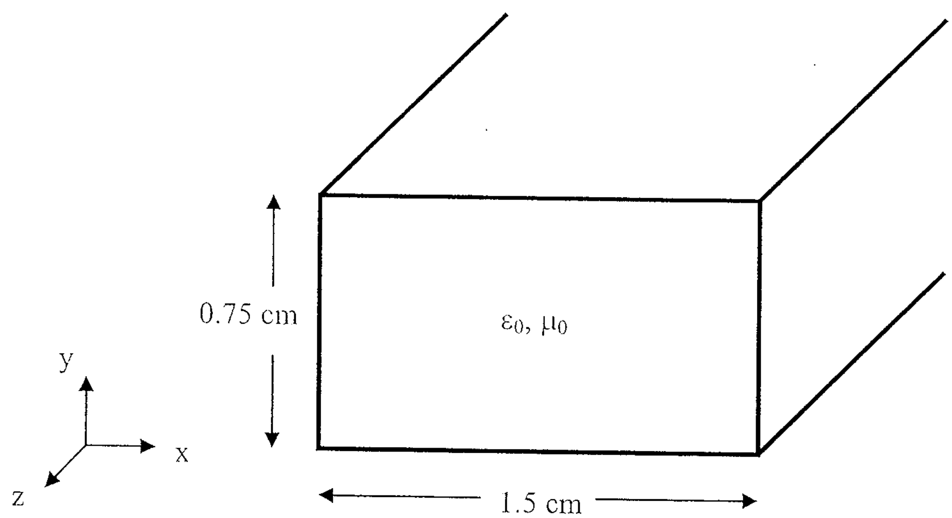

Figure 2: Hollow waveguide with perfectly conducting walls (for problem 3).

Maxwell's Equations, in differential and integral form are: Other potentially useful relationships are:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \oint_C \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{s} \qquad \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{s} \qquad \vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right) = \mu \vec{H}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \oint_S \vec{D} \cdot d\vec{s} = Q \qquad \lambda = \frac{2\pi}{\beta} = \frac{u}{f} = \frac{1}{f\sqrt{\mu\varepsilon}}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \oint_S \vec{B} \cdot d\vec{s} = 0 \qquad \varepsilon_0 = 8.854 \times 10^{-12} \, F/m$$

$$\mu_0 = 4\pi \times 10^{-7} \, H/m$$

Vector Differential Relationships in Cylindrical Coordinates (r, ϕ, z)

$$\vec{\nabla} V = \hat{a}_r \frac{\partial V}{\partial r} + \hat{a}_\phi \frac{\partial V}{r \partial \phi} + \hat{a}_z \frac{\partial V}{\partial z}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{\partial A_\phi}{r \partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla} \times \vec{A} = \hat{a}_r \left(\frac{\partial A_z}{r \partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{a}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \hat{a}_z \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\phi) - \frac{\partial A_r}{\partial \phi} \right]$$

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Divergence Thm: $\int_V \vec{\nabla} \cdot \vec{A} dv = \oint_S \vec{A} \cdot d\vec{s}$

Stokes Thm: $\int_V \vec{\nabla} \times \vec{A} \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{\ell}$

For lossless transmission lines:

$$Z_{in} = R_o \frac{Z_L + jR_o \tan\beta\ell}{R_o + jZ_L \tan\beta\ell}$$