1. **(25 Points)** Let $\mathbf{X}$ and $\mathbf{Y}$ be two independent identically distributed random variables taking on values in $\mathbf{N}$ (the natural numbers) with

$$P(\{\mathbf{X}=i\}) = P(\{\mathbf{Y}=i\}) = \frac{1}{2^i}, \quad i = 1, 2, 3, \ldots.$$

   (a) Find $P(\{\min(\mathbf{X}, \mathbf{Y}) = k\})$, for $k \in \mathbf{N}$.
   (b) Find $P(\{\mathbf{X} = \mathbf{Y}\})$.
   (c) Find $P(\{\mathbf{Y} > \mathbf{X}\})$.
   (d) Find $P(\{\mathbf{Y} = k\mathbf{X}\})$ for a given natural number k.

2. **(25 Points)** Let $\{\mathbf{X}_n\}_{n\geq 1}$ be a sequence of binomially distributed random variables, with the n-th random variable $\mathbf{X}_n$ having pmf

$$p_{\mathbf{X}_n}(k) = P(\{\mathbf{X}_n = k\}) = \binom{n}{k} p_n^k (1-p_n)^{n-k}, \qquad k = 0, \ldots n, \qquad p_n \in (0,1).$$

   Show that, if the p_n have the property that $np_n \to \lambda$ as $n \to \infty$, where λ is a positive constant, then the sequence $\{\mathbf{X}_n\}_{n\geq 1}$ converges in distribution to a Poisson random variable $\mathbf{X}$ with mean λ.

   Hint: You may find the following fact useful:

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

3. **(25 Points)** Let $\mathbf{X}(t)$ be a real Gaussian random process with mean function $\mu(t)$ and autocovariance function $C_{XX}(t_1, t_2)$.
   (a) Write the expression for the n-th order characteristic function of $\mathbf{X}(t)$ in terms of $\mu(t)$ and $C_{XX}(t_1, t_2)$.
   (b) Show that the probabilistic description of $\mathbf{X}(t)$ is completely characterized by $\mu(t)$ and autocovariance function $C_{XX}(t_1, t_2)$.
   (c) Show that if $\mathbf{X}(t)$ is wide-sense stationary then it is also strict-sense stationary.

4. **(25 Points)** Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots$ be a sequence of independent, identically distributed random variables, each having Cauchy pdf

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

   Let

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i.$$

   Find the pdf of Y_n. Describe how the pdf of $\mathbf{Y}_n$ depends on n. Does the sequence $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \ldots$ converge in distribution? If yes, what is the distribution of the random variable it converges to?