

Part I: Image Processing [40 points]

Consider the continuous 2D signal, $f(x, y)$. Sampling of this signal may be (ideally) represented as multiplication by the *comb* function:

$$\delta_s(x, y; \Delta x, \Delta y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$

In this case, the sampled signal is

$$f_s(x, y) = f(x, y)\delta_s(x, y; \Delta x, \Delta y)$$

- (1) (20 points) Derive $F_s(u, v)$, the 2D Fourier transform of $f_s(x, y)$.
- (2) (20 points) Let $f(x, y)$ be frequency band-limited to $u < U$ and $v < V$. Derive and graphically illustrate the Nyquist sampling criteria related to Δx and Δy .

Part II: Nuclear Magnetic Resonance [60 points]

Magnetic resonance imaging (MRI) relies upon contrasts generated by three physical properties of the tissue to be imaged: (1) proton density, (2) spin-lattice relaxation time (T_1), and (3) spin-spin relaxation time (T_2).

- (3) (10 points) Briefly explain spin-lattice (T_1) relaxation.
- (4) (10 points) Briefly explain spin-spin (T_2) relaxation.

For questions (5)–(7), consider a (finite) object having proton density $\rho(x, y)$ such that $\int_x \int_y \rho(x, y) dx dy = S_0$. Assume that the relaxation times and magnetic environment of the object are spatially uniform — i.e., $T_1(x, y) = T_1$, $T_2(x, y) = T_2$, $\omega_0(x, y) = \gamma B_0(x, y) = \gamma B_0 = \omega_0$, with γ corresponding to the gyromagnetic ratio of the nucleus being investigated.

- (5) (10 points) At $t = 0$, a 90 degree radio frequency (RF) tip is applied (uniformly) to the object to bring the magnetization vectors into the transverse plane. Provide an equation for the received free induction decay (FID) signal as a function of time, $S(t)$.
- (6) (15 points) Now a two-pulse (saturation recovery) experiment is conducted beginning with a 90 degree RF tip at $t = 0$, followed by a second 90 degree RF tip at $t = t_s^-$. What is the amplitude of the transverse magnetization signal (i.e., that along the x-axis) measured at $t = t_s^+$?
- (7) (15 points) The same experiment as in (6) is conducted, but using an α degree RF tip at $t = t_s$. Provide an equation for the FID signal (transverse magnetization) observed subsequent to this 2nd tip — i.e., provide an equation for $S(t), t > t_s$.