LTI and LT Systems – State-Space Approach

Unless otherwise stated, you need to justify your answers.

1. (25 points) For a given matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix},$$

compute e^{At}.

2. (25 points) Consider a LTI system $\dot{x} = Ax + Bu$, $y = Cx$, given specifically by

$$\dot{x} = \begin{bmatrix} -3 & 1 & & & & \\ & -3 & 1 & & & \\ & & -3 & & & \\ & & & 0 & & \\ & & & & -3 & 1 \\ & & & & & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ \alpha & 2 \\ 0 & \beta \\ -3 & 5 \\ 4 & -1 \end{bmatrix} u \tag{1}$$

$$y = \begin{bmatrix} \gamma & 0 & 1 & \eta & 2 & 1 \end{bmatrix} x,$$

where the system matrices are dependent on the parameters α, β, γ, and η.

(a) (5 points) Suppose there is no external input, i.e., $u \equiv 0$. Determine the stability of the resulting autonomous system $\dot{x} = Ax$.

(b) (5 points) What are the conditions on the parameters α, β, γ, and η so that the original system (1) is controllable (if possible)?

(c) (5 points) What are the conditions on the parameters α, β, γ, and η so that the original system (1) is observable (if possible)?

(d) (5 points) What are the conditions on the parameters α, β, γ, and η so that the original system (1) is minimal (if possible)?

(e) (5 points) What are the conditions on the parameters α, β, γ, and η so that the original system (1) is stabilizable (if possible)? Recall that by system (1) being stablizable, we mean that there exists a state feedback control $u = -Kx$ so that the closed loop system $\dot{x}_c = (A - BK)x_c$ is stable.

3. (30 points) Consider two discrete-time LTI systems

$$x[k+1] = A_1 x[k] + B_1 u[k] = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]. \tag{2}$$

$$x[k+1] = A_2 x[k] + B_2 u[k] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u[k]. \tag{3}$$

(a) (5 points) Suppose that system (2) starts from zero initial condition $x[0] = 0$. Find the reachable subspace $\mathcal{R}_k$ of system (2) in k steps for large k.

(b) (5 points) Suppose that system (3) starts from zero initial condition $x[0] = 0$. Find the reachable subspace $\mathcal{R}_k$ of system (3) in k steps for large k.

(c) (10 points) Now consider a discrete-time LTV system given by

$$x[k+1] = A[k]x[k] + B[k]u[k], \tag{4}$$

where

$$A[k] = \begin{cases} A_1 & k = 0, 1, \\ A_2 & k = 2, 3, 4, \dots. \end{cases}$$

In other words, system (4) first follows system (2) for the first two time steps, and then follows system (3) for the rest of the time. Find the reachable subspace $\mathcal{R}_k$ of system (4) in k steps for large k, starting from $x[0] = 0$.

(d) (10 points) Next consider the discrete-time LTV system

$$x[k+1] = A[k]x[k] + B[k]u[k], \tag{5}$$

where

$$A[k] = \begin{cases} A_2 & k = 0, 1, \\ A_1 & k = 2, 3, 4, \dots. \end{cases}$$

This system is very similar to system (4), with the only difference being that the order that systems (2) and (3) are followed is reversed. For this system (5), starting from $x[0] = 0$, find its reachable subspace $\mathcal{R}_k$ in k steps for large k.

4. (20 points) Suppose that a LTV system is given by

$$\dot{x}(t) = A(t)x(t) = \begin{bmatrix} \cos t & 2\cos t \\ 3\cos t & 0 \end{bmatrix} x(t),$$

where $x(t) \in \mathbb{R}^2$.

(a) (10 points) Find the state transition matrix $\Phi(t, \tau)$ of the system. You may want to use the result of Problem 1 to simplify your solution.

(b) (5 points) Is the system asymptotically stable?

(c) (5 points) Starting from arbitrary $x(0)$, will the system trajectory $x(t)$ stay bounded as $t \to \infty$?