Problem 1. [30 pts] An upsampler system is shown in Figure 1. For all parts, consider the ideal case where \( h[n] = \frac{\sin(\frac{\pi}{2} n)}{\frac{\pi}{2} n} \) AND the input signal to this system is the sampled signal \( x[n] = x_a(nT_s) \) where \( T_s = \frac{2\pi}{40} \) and

\[
x_a(t) = \left(\frac{2\pi}{40}\right)^2 \frac{\sin(15t) \sin(5t)}{\pi t}
\]

Figure 1.

(a) For the polyphase filter \( h_0[n] = h[3n] \), determine an expression for the output, denoted \( y_0[n] = x[n] \ast h_0[n] \).

(b) For the polyphase filter \( h_1[n] = h[3n + 1] \), determine an expression for the output, denoted \( y_1[n] = x[n] \ast h_1[n] \).

(c) For the polyphase filter \( h_2[n] = h[3n + 2] \), determine an expression for the output, \( y_2[n] = x[n] \ast h_2[n] \).

(d) Determine an expression for the overall output of this system, denoted \( y[n] \).

Problem 2. [40 pts] A second-order digital filter is to be designed from an analog filter having two poles in the s-plane at \( p_1 = -1 + 2j \) and \( p_2 = -1 - 2j \) and two zeros at \( z_1 = j \) and \( z_2 = -j \),

\[
H_a(s) = \frac{G(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)}
\]

via the bilinear transformation method characterized by the mapping

\[
s = \frac{z - 1}{z + 1}
\]

(a) Is the resulting digital filter (BIBO) stable? Briefly explain why or why not.

(b) Denote the frequency response of the resulting digital filter as \( H(\omega) \) (the DTFT of its impulse response). You are given that in the range \( 0 < \omega < \pi \), there is only one value of \( \omega \) for which \( H(\omega) = 0 \). Determine that value of \( \omega \).

(c) Draw a pole-zero diagram for the resulting digital filter. Give the exact locations of the poles and zeros of the digital filter in the z-plane.

(d) Plot the magnitude of the DTFT of the resulting digital filter, \(|H(\omega)|\), over \(-\pi < \omega < \pi\). You are given that \( H(0) = 0.8 \). Be sure to indicate any frequency for which \( |H(\omega)| = 0 \). Also, specifically note the numerical value of \( |H(\omega)| \) for \( \omega = \frac{\pi}{3} \) and \( \omega = \pi \).

(e) Determine the difference equation for the resulting digital filter.
Problem 3. [30 pts] Consider the length-7 DT signal below where the first value corresponds to \( n = 0 \).

\[
x[n] = \{ +1, +1, +1, -1, -1, +1, -1 \}
\]

One way to compute autocorrelation is \( r_{xx}[n] = x[n] \ast x^*[-n] \). Suppose we want to use DFT-based processing to compute autocorrelation. One issue is that the DFT assumes that signals start at \( n = 0 \). So, let’s time-shift the time-reversed signal to the right by 7-1=6 to form \( x[-(n-6)] \) so that it starts at \( n = 0 \); \( h[n] = x[-(n-6)] \) is the causal matched filter. Since \( x[n] \) is real-valued,

\[
h[n] = x[-(n-6)] \quad \text{DTFT} \quad H(\omega) = X^*(\omega)e^{-j6\omega}
\]

(a) A 16-point DFT of \( x[n] \) is computed and denoted as \( X_{16}(k) \), \( k = 0, 1, \ldots 15 \). We then form \( Y_{16}(k) \) via the point-wise product defined below

\[
Y_{16}(k) = X_{16}(k)X^*_{16}(k)e^{-j\frac{2\pi k}{16}} \quad k = 0, 1, \ldots, 15
\]

Finally, compute \( y_{16}[n] \) as the 16-point Inverse DFT of \( Y_{16}(k) \). Determine and list all 16 values of \( y_{16}[n] \), for \( n = 0, 1, \ldots, 15 \). Show all work.

(b) A 13-point DFT of \( x[n] \) is computed and denoted as \( X_{13}(k) \), \( k = 0, 1, \ldots 12 \). We then form \( Y_{13}(k) \) via the point-wise product defined below

\[
Y_{13}(k) = X_{13}(k)X^*_{13}(k)e^{-j\frac{2\pi k}{13}} \quad k = 0, 1, \ldots, 12
\]

Finally, compute \( y_{13}[n] \) as the 13-point Inverse DFT of \( Y_{13}(k) \). Determine and list all 13 values of \( y_{13}[n] \), for \( n = 0, 1, \ldots, 12 \).

(c) An 10-point DFT of \( x[n] \) is computed and denoted as \( X_{10}(k) \), \( k = 0, 1, \ldots, 9 \). We then form \( Y_{10}(k) \) via the point-wise product defined below

\[
Y_{10}(k) = X_{10}(k)X^*_{10}(k)e^{-j\frac{2\pi k}{10}} \quad k = 0, 1, \ldots, 9
\]

Finally, compute \( y_{10}[n] \) as the 10-point Inverse DFT of \( Y_{10}(k) \). Determine and list all 10 values of \( y_{10}[n] \), for \( n = 0, 1, \ldots, 9 \).