Problem 1. (40 pts) Let $x_1[n]$ and $x_2[n]$ be DT signals with autocorrelations and cross-correlations defined in terms of convolution as below.

$$
\begin{align*}
    r_{x_1x_1}^{*}[\ell] &= x_1[\ell] \ast x_2^*[\ell] \\
    r_{x_2x_2}^{*}[\ell] &= x_2[\ell] \ast y_2^*[\ell] \\
    r_{x_1x_2}^{*}[\ell] &= x_1[\ell] \ast x_2^*[\ell] \\
    r_{x_2x_1}^{*}[\ell] &= x_2[\ell] \ast x_1^*[\ell]
\end{align*}
$$

(a) Determine the autocorrelation $r_{x_1x_1}^{*}[\ell]$ of the length-3 sequence $x_1[n]$ below, which is written two different ways. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$
x_1[n] = \{1, 1, -1\} = \delta[n] + \delta[n - 1] - \delta[n - 2]
$$

(b) Determine the autocorrelation $r_{x_2x_2}^{*}[\ell]$ of the length-5 sequence $x_2[n]$ below, which is written two different ways. Write your answer in sequence form indicating where the value for $\ell = 0$ is located.

$$
x_2[n] = \{1, 1, 1, -1, 1\} = \delta[n] + \delta[n - 1] + \delta[n - 2] - \delta[n - 3] + \delta[n - 4]
$$

(c) The sequence $x_1[n]$ defined above is input to the system described by the simple difference equation below. Do a stem plot of the cross-correlation $r_{y_1y_1}^{*}[\ell]$ between the output and input.

$$
y_1[n] = 4x_1[n - 3] + x_1[n - 4]
$$

(d) The sequence $x_2[n]$ defined above is input to the same system described by the simple difference equation below. Do a stem plot of the cross-correlation $r_{y_2y_2}^{*}[\ell]$ between the output and input.

$$
y_2[n] = 4x_2[n - 3] + x_2[n - 4]
$$

(e) Sum your answers to parts (c) and (d) to form the sum below. Do a stem plot of $r_{y_2}^{*}[\ell]$.

$$
r_{y2}^{*}[\ell] = r_{y_1y_1}^{*}[\ell] + r_{y_2y_2}^{*}[\ell]
$$
Problem 2. (20 pts)

(a) Consider \(h[n]\) to be an all-pass filter with respective impulse response below, that is, \(|H(\omega)| = 1\) for all \(\omega\).

\[
h[n] = \frac{1}{p} \left\{ \delta[n] + (p^2 - 1)p^n u[n] \right\}
\]

Is the product

\[
g[n] = e^{j\omega n} h[n]
\]

an all-pass filter for any and all values of the frequency \(\omega\)? Explain your answer. Your explanation is much more important than your answer.

(b) Consider \(h_1[n]\) and \(h_2[n]\) to be two distinct all-pass filters \((p_1 \neq p_2)\) with respective impulse responses below. Is the sum \(h[n] = h_1[n] + h_2[n]\) an all-pass filter for any and all values of \(p_1\) and \(p_2\) \((p_1 \neq p_2)\)? Explain your answer. Your explanation is much more important than your answer.

\[
h_1[n] = \frac{1}{p_1} \left\{ \delta[n] + (p_1^2 - 1)p_1^n u[n] \right\}
\]

\[
h_2[n] = \frac{1}{p_2} \left\{ \delta[n] + (p_2^2 - 1)p_2^n u[n] \right\}
\]
Problem 3. [40 points]

(a) Consider two LTI systems connected in SERIES. System 1 has impulse response $h_1[n]$ below, where $p_1 = 0.5$ (in fractional form $p_1 = \frac{1}{2}$)

$$h_1[n] = \frac{1}{p_1} \left\{ \delta[n] + (p_1^2 - 1)p_1^n u[n] \right\}$$

System 2 has impulse response $h_2[n]$ below, where $p_2 = -0.5$

$$h_2[n] = \frac{1}{p_2} \left\{ \delta[n] + (p_2^2 - 1)p_2^n u[n] \right\}$$

(b) Determine a closed-form expression for the impulse response, $h[n]$, of the overall system. Write how $h[n]$ is related to $h_1[n]$ and $h_2[n]$, and then show all work in determining your final answer. You may want to make use of the convolution result below. Note: you can solve the rest of this problem (except the extra credit) without the answer to part (a.)

$$\alpha^n u[n] * \beta^n u[n] = \frac{\alpha}{\alpha - \beta} \alpha^n u[n] + \frac{\beta}{\beta - \alpha} \beta^n u[n]$$

(c) Determine the Z-Transform of the overall system. Draw the pole-zero diagram.

(d) Plot the magnitude of the frequency response $|H(\omega)|$ of the overall system over $-\pi < \omega < \pi$. Explain your answer.

(e) Write the difference equation for the overall system.

(f) Determine the output $y[n]$ of the overall system when the input is the sum of sinewaves (turned on forever) below

$$x[n] = 4 + 3(j)^n + 2(-j)^n + (-1)^n$$

(g) Consider creating a new impulse response from your answer for part (a) as

$$g[n] = h[2n + 1] \quad \text{where: } h[n] \quad \text{is the overall impulse response of the system}$$

That is, $g[n]$ is formed by keeping only the values of $h[n]$ for odd values of time, i.e., throwing the values of $h[n]$ for even values of time. Compute the energy of $g[n]$:

$$E_g = \sum_{n=-\infty}^{\infty} g^2[n]$$