LTI and LT Systems – State-Space Approach

Note: Unless otherwise stated, you need to justify your answers to get the full credit.

Problem 1. (20 points) A system with input $u$ and output $y$ is given by
\[ \ddot{y} + 2y\dot{y} + uy = 2u. \]
(a) (5 pts) Identify a set of state variables and construct a state space model of the system.
(b) (5 pts) Assuming $u = 2$ is constant, find all the equilibrium points of the system.
(c) (10 pts) Under the same assumption as in (b), determine the local stability of the equilibrium point (or points) you found in (b).

Problem 2. (15 points) The following discrete-time system is given
\[ x[k+1] = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} x[k] \]
\[ y[k] = \begin{bmatrix} -1 & 1 \end{bmatrix} x[k]. \]
Note that the matrix $A$ is nilpotent: $A^2 = 0$.
(a) (10 pts) Suppose only the outputs $y[0]$ and $y[1]$ are known: $y[0] = 1$; $y[1] = 0$. Can we uniquely determine the initial state $x[0]$? If yes, find $x[0]$; if no, explain why.
(b) (5 pts) Suppose only the outputs $y[1]$ and $y[2]$ are known: $y[1] = 1$; $y[2] = 0$. Can we uniquely determine the initial state $x[0]$? If yes, find $x[0]$; if no, explain why.

Problem 3. (45 pts) Consider the following LTI system:
\[ \dot{x} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} -1 & 1 \end{bmatrix} x. \]  \hspace{1cm} (1)
Note that the matrix $A$ is nilpotent: $A^2 = 0$.
(a) (7 pts) Find all the eigenvalues and eigenvectors of $A$ as well as its Jordan canonical form.
(b) (6 pts) Compute $e^{At}$ for $t \geq 0$.
(c) (6 pts) For the autonomous system $\dot{x} = Ax$, determine its stability and find, if possible, all the initial states $x(0)$ starting from which the solution $x(t)$ remains bounded for all $t \geq 0$. 

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(d) (6 pts) For the system \( \dot{x} = Ax + Bu \), starting from \( x(0) = 0 \), can you find a control \( u(t) \) for \( t \in [0, 1] \) so that \( x(1) = [-4, -2]^T \)? Find such a control \( u(t) \) explicitly if the answer is yes and explain why if the answer is no. What if the goal is to have \( x(1) = [1, 2]^T \)?

(e) (6 pts) For the system \( \dot{x} = Ax + Bu \), can you find a state feedback controller \( u = -Kx \) under which the closed-loop system \( \dot{x} = A_{cl}x \) has solutions \( x(t) \to 0 \) as \( t \to \infty \) regardless of \( x(0) \)? Find such a \( K \) if the answer is yes and explain why if the answer is no. What if the goal is to have all the solutions \( x(t) \) be bounded for all \( t \)? Find the condition on \( K \) if the latter goal is achievable.

(f) (6 pts) Is the system \( \dot{x} = Ax, y = Cx \), observable? What is its unobservable subspace? Is it detectable?

(g) (4 pts) Find the transfer function \( H(s) = \frac{Y(s)}{U(s)} \) of the system (1). Is it BIBO stable?

(h) (4 pts) Suppose an output feedback controller \( u = -ay \) is adopted in system (1). Can you find a gain \( \alpha \in \mathbb{R} \) so that the resulting closed-loop system satisfies that \( y(t) \to 0 \) as \( t \to \infty \) for all \( x(0) \)? Find such an \( \alpha \) if the answer is yes and explain why if the answer is no.

**Problem 4. (20 points)** Consider a LTV system \( \dot{x}(t) = A(t)x(t) \) where \( x(t) \in \mathbb{R}^2 \) and

\[
A(t) = \begin{bmatrix}
-2 - 6t & 4 + 8t \\
-1 - 2t & 2 + 2t
\end{bmatrix} = \begin{bmatrix}
-2 & 4 \\
-1 & 2
\end{bmatrix} + 2t \cdot \begin{bmatrix}
-3 & 4 \\
-1 & 1
\end{bmatrix}, \quad \forall t \geq 0.
\]

Note that \( A_1 \) is the exact same state dynamics matrix in Problem 3 (hence \( A_1^2 = 0 \)) and \( A_2 = A_1 - I \). Find the fundamental matrix \( \Phi(t) \), namely, the state transition matrix \( \Phi(t, 0) \), of the system.