Problem 1. [30 points]

(a) Consider the discrete-time signal below.

\[ x[n] = (0.5)^n e^{j\frac{\pi}{4}n} \{u[n] - u[n - 5]\} \]

Determine a closed-form expression for the DTFT, \( X(\omega) \), of \( x[n] \). Show all work.

(b) Show how your answer to 1(a) changes for the discrete-time signal below.

\[ y[n] = (0.5)^n e^{j\frac{\pi}{8}n} \{u[n - 2] - u[n - 7]\} \]

Determine a closed-form expression for the DTFT, \( Y(\omega) \), of \( y[n] \).

(c) The damped sinusoidal signal \( x_a(t) = e^{-4ln(2)t} e^{j2\pi t} \{u(t) - u(t - 1)\} \) is sampled every \( T_s = 0.25 \) seconds to form \( x[n] = x_a(nT_s) \), where, again, \( T_s \) is a quarter of a second. Determine a closed-form expression for the DTFT \( X(\omega) \) of the \( x[n] \) thus obtained. 

Hint: \( ln(2) \) equal to the natural logarithm of 2 was chosen to make the numbers work out nicely; same with the factor of 4 in the exponent. Recall \( e^{ln(x)} = x \).

\[ x[n] = x_a(nT_s) \quad \text{where: } T_s = 0.25 \text{ secs} \quad \text{and} \quad x_a(t) = e^{-4ln(2)t} e^{j2\pi t} \{u(t) - u(t - 1)\} \]

Problem 2. [20 points] Consider a causal FIR filter of length \( M = 9 \) with impulse response as defined below:

\[ h_p[n] = \sum_{\ell=-\infty}^{\infty} \frac{\sin \left[ \frac{\pi}{2} \left( n + \frac{3}{2} + \ell \right) \right]}{\pi \left( n + \frac{3}{2} + \ell \right)} \{u[n] - u[n - 9]\} \]

(a) Determine the 9-pt DFT of \( h_p[n] \), denoted \( H_9(k) \), for \( 0 \leq k \leq 8 \). You can EITHER write an expression for \( H_9(k) \), OR list the numerical values: \( H_9(0) = ?, H_9(1) = ?, H_9(2) = ?, H_9(3) = ?, H_9(4) = ?, H_9(5) = ?, H_9(6) = ?, H_9(7) = ?, H_9(8) = ? \).

(b) Consider the sequence \( x[n] \) of length \( L = 9 \) below, equal to a sum of 9 finite-length sinewaves.

\[ x[n] = \sum_{k=0}^{8} e^{j\frac{2\pi}{9}k} \{u[n] - u[n - 9]\} \]

\( y_9[n] \) is formed by computing \( X_9(k) \) as an 9-pt DFT of \( x[n] \), \( H_9(k) \) as a 9-pt DFT of \( h[n] \) and, finally, then \( y_9[n] \) is computed as the 9-pt inverse DFT of \( Y_9(k) = X_9(k)H_9(k) \).

Express the result \( y_9[n] \) as a weighted sum of finite-length sinewaves similar to how \( x[n] \) is written above.
Problem 3. [50 points] Let $x[n]$ and $y[n]$ be DT signals with autocorrelations and cross-correlations defined in terms of convolution as below.

$$
r_{xx}[\ell] = x[\ell] * x^*[-\ell] \quad r_{yy}[\ell] = y[\ell] * y^*[-\ell] \quad r_{xy}[\ell] = x[\ell] * y^*[-\ell] \quad r_{yx}[\ell] = y[\ell] * x^*[-\ell]$$ (1)

(a) Consider the case where $x[n]$ and $y[n]$ are both causal, finite-length signals of duration $N$. That is, $x[n]$ and $y[n]$ are both only nonzero for $n = 0, 1, ..., N - 1$. A concatenated signal of length $2N$ is formed as below:

$$z[n] = x[n] + y[n - N]$$ (2)

Express the autocorrelation, $r_{zz}[\ell]$, for $z[n]$ in terms of $r_{xx}[\ell]$, $r_{yy}[\ell]$, $r_{xy}[\ell]$, and $r_{yx}[\ell]$.

(b) Consider a case where $x[n]$ & $y[n]$ form a complementary pair of +1's and -1's satisfying

$$r_{xx}[\ell] + r_{yy}[\ell] = 2N\delta[\ell]$$ (3)

Simplify your answer for $r_{zz}[\ell]$ in part (a) for this special case.

(c) Consider the Barker codes of length 4 below, where the first value corresponds to $n = 0$.

$$x[n] = \{1, 1, 1, -1\} \quad y[n] = \{1, 1, -1, 1\}$$ (4)

such that $z[n] = x[n] + y[n - N]$ is $z[n] = \{1, 1, 1, -1, 1, 1, -1, 1\}$. Determine the autocorrelation $r_{zz}[\ell]$ for $z[n]$ using the results that you derived above (you can compare to a direct calculation of $r_{zz}[\ell]$ to check your answer.)

(d) Repeat the steps above for the case where the concatenated signal of length $2N$ is formed as below with a negative sign on the second term.

$$w[n] = x[n] - y[n - N]$$ (5)

Express the autocorrelation, $r_{ww}[\ell]$, for $w[n]$ in terms of $r_{xx}[\ell]$, $r_{yy}[\ell]$, $r_{xy}[\ell]$, and $r_{yx}[\ell]$.

(e) Consider a case where $x[n]$ & $y[n]$ form a complementary pair of +1's and -1's satisfying

$$r_{xx}[\ell] + r_{yy}[\ell] = 2N\delta[\ell]$$ (6)

Simplify your answer for $r_{ww}[\ell]$ in part (d) for this special case.

(f) Again, let $x[n] = \{1, 1, 1, -1\}$ and $y[n] = \{1, 1, -1, 1\}$ be Barker codes of length 4, such that

$$w[n] = x[n] - y[n - N] = \{1, 1, 1, -1, 1, 1, -1, 1\}$$ (7)

Determine the autocorrelation $r_{ww}[\ell]$ for $w[n]$ using the results derived above.

(g) Sum your answers to parts (b) and (e) to determine $r_{ww}[\ell] = r_{xx}[\ell] + r_{ww}[\ell]$. Do a stem plot of the sum $r_{ww}[\ell]$. Compare against the sum of your answers to (c) and (f).