1. (20 pts) Solve the following linear program,

$$\text{maximize} \quad -x_1 - 2x_2 + 4x_3$$

$$\text{subject to} \quad x_1 + 2x_2 - x_3 = 5$$
$$2x_1 + 3x_2 - x_3 = 6$$
$$x_1 \text{ free, } x_2 \geq 0, \ x_3 \leq 0.$$ 

2. (20 pts) Formulate the first-order necessary conditions for the quadratic program,

$$\text{minimize} \quad \frac{1}{2} x^\top Q x - b^\top x$$

$$\text{subject to} \quad Ax = c,$$

where $x \in \mathbb{R}^n$, $c \in \mathbb{R}^m$, $m \leq n$, and $Q = Q^\top \succ 0$.

- (15 pts) Represent the obtained conditions as a system of linear equations and write down the solution to the problem.

- (5 pts) What is the condition, involving $A$ and $Q$, that must be satisfied for the solution to be unique?

3. (20 pts) Consider the optimization problem,

$$\text{maximize} \quad -x_1^2 + x_1 - x_2 - x_1 x_2$$

$$\text{subject to} \quad x_1 \geq 0, \ x_2 \geq 0.$$
(i) \((10\text{ pts})\) Characterize feasible directions at the point

\[
x^* = \begin{bmatrix}
\frac{1}{2} \\
0
\end{bmatrix}.
\]

(ii) \((10\text{ pts})\) Write down the second-order necessary condition for \(x^*\). Does the point \(x^*\) satisfy this condition?

4. \((20\text{ pts})\) Consider the following primal problem:

maximize \(x_1 + 2x_2\)

subject to

\[-2x_1 + x_2 + x_3 = 2\]
\[-x_1 + 2x_2 + x_4 = 7\]
\[x_1 + x_5 = 3\]
\[x_i \geq 0, \; i = 1, 2, 3, 4, 5.\]

(i) \((5\text{ pts})\) Construct the dual problem corresponding to the above primal problem.

(ii) \((15\text{ pts})\) It is known that the solution to the above primal is \(x^* = \begin{bmatrix} 3 & 5 & 3 & 0 & 0 \end{bmatrix}^T\).

Find the solution to the dual.

5. \((20\text{ pts})\) Find the minimizer of

\[f(x_1, x_2) = \frac{1}{2}x_1^2 + x_2^2 + x_1 + \frac{1}{2}x_2 + 3\]

using the conjugate gradient algorithm. The starting point is \(x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T\).