Note: Unless otherwise stated, you need to justify your answers to get the full credit.

**Problem 1. (40 points)** Consider the LTI system \( \dot{x} = Ax + Bu, \ y = Cx, \) where
\[
A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 0 & 1 \\ -2 & -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.
\]

(a) (4 pts) Compute \( e^{At} \) for \( t \geq 0 \).

(b) (4 pts) For the autonomous system \( \dot{x} = Ax \), find its three modes and determine its stability.

(c) (4 pts) Find the set of all possible initial states \( x(0) \in \mathbb{R}^3 \) starting from which the solutions to \( \dot{x} = Ax \) satisfy \( x(t) \to 0 \) as \( t \to \infty \). If instead it is desired that \( x(t) \) remains bounded for all \( t \geq 0 \), what is the set of all possible \( x(0) \)?

(d) (4 pts) Is the given LTI system controllable? Find the reachable subspace (controllable subspace) of the system.

(e) (4 pts) Is the given LTI system observable? What is its unobservable subspace?

(f) (4 pts) Using the state feedback control \( u = -Kx \), can you find a proper gain matrix \( K \in \mathbb{R}^{1 \times 3} \) so that the resulting closed-loop system \( \dot{x} = A_d x \) is stable?

(g) (4 pts) Using the state feedback control \( u = -Kx \), can you find a proper gain matrix \( K \in \mathbb{R}^{1 \times 3} \) so that the resulting closed-loop system \( \dot{x} = A_d x, \ y = Cx \), satisfies \( y(t) \to 0 \) as \( t \to \infty \) for all \( x(0) \in \mathbb{R}^3 \)? Find such a \( K \) if the answer is yes; otherwise, state your reasons.

(h) (4 pts) Can you find \( L \in \mathbb{R}^3 \) such that \( A - LC \) is stable? More generally, what are the possible sets of eigenvalues of \( A - LC \) for arbitrary choices of \( L \).

(i) (4 pts) Find the transfer function \( H(s) = \frac{Y(s)}{U(s)} \). Is the system BIBO stable, i.e., for \( x(0) = 0 \) and bounded input \( u(t) \), the system output \( y(t) \) will remain bounded for all \( t \geq 0 \)?

(j) (4 pts) Suppose \( x(0) = \begin{bmatrix} -2 & -4 & 4 \end{bmatrix}^T \) and \( u(t) = 1 \) for all \( t \geq 0 \). Find \( y(t) \) for \( t \geq 0 \) for the given LTI system.

**Problem 2. (20 points)** To the extent possible, find the fundamental matrix \( \Phi(t) \) of the following LTV system
\[
\dot{x}(t) = \begin{bmatrix} -t & 1 \\ 0 & -1 \end{bmatrix} x(t).
\]
Problem 3. (20 points) A discrete-time LTI system is given as

\[ x[k + 1] = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k], \quad k = 0, 1, \ldots. \]

From the initial state \( x[0] = [1 \quad 1]^T \), find the control input \( u[0], u[1], \) and \( u[2] \) that can steer the state to zero at time \( k = 3 \) (i.e., \( x[3] = [0 \quad 0]^T \)) with the least control energy \( |u[0]|^2 + |u[1]|^2 + |u[2]|^2 \).

Problem 4. (20 points) Find all the equilibrium points of the following nonlinear system and determine the local stability around each of them, if possible:

\[
\begin{align*}
\dot{x}_1 &= (x_1^2 - 1)(x_2 - 2) \\
\dot{x}_2 &= -x_2(x_1^2 + 1).
\end{align*}
\]