Problem 1. (50pt)
Consider an X-ray imaging system shown in the figure below.

Photon are emitted from an X-ray source and collimated by a pin hole in a lead shield. The collimated X-rays then pass in a straight line through an object of length $T$ with density $u(x)$ where $x$ is the depth into the object. The number of photons in the beam at depth $x$ is denoted by the random variable $Y_x$ with Poisson density given by

$$P\{Y_x = k\} = \frac{e^{-\lambda_x} \lambda_x^k}{k!},$$

where $x$ is measured in units of cm and $\mu(x)$ is measured in units of cm$^{-1}$.

a) (10pt) Calculate the $E\{Y_x\}$.

b) (10pt) Write a differential equation which describes the behavior of $\lambda_x$ as a function of $x$.

c) (10pt) Calculate an expression for $\lambda_x$ in terms of $u(x)$ and $\lambda_0$ by solving the differential equation.

d) (10pt) Calculate an expression for the integral of the density, $\int_0^T u(x)dx$, in terms of $\lambda_0$ and $\lambda_T$.

e) (10pt) Give an estimate for the integral of the density, $\int_0^T u(x)dx$, in terms of the measured values of $Y_0$ and $Y_T$.

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Problem 2. (50pt)
Consider the 2D difference equation

$$y(m, n) = bx(m, n) + ay(m - 1, n) + ay(m, n - 1) - a^2 y(m - 1, n - 1)$$

where $b \in \mathbb{R}$ and $a \in (-1, 1)$ are two constants, and $Y(z_1, z_2)$ and $X(z_1, z_2)$ are the 2D Z-transforms of $y(m, n)$ and $x(m, n)$ respectively.

a) (10pt) Calculate $H(z_1, z_2) = \frac{Y(z_1, z_2)}{X(z_1, z_2)}$, the 2D transfer function of the causal system. Make sure to express your result in factored form.

b) (10pt) Calculate $h(m, n)$, the impulse response of the system with transfer function $H(z_1, z_2)$.

c) (10pt) In an application, $x(m, n)$ is an input image; and $y(m, n)$ is an output filtered image. Specify a relationship between $a$ and $b$ so that the average values of the input and output images remain the same.

d) (10pt) For parts d) and e), assume the input, $x(m, n)$, are i.i.d. Gaussian random variables with mean zero and variance 1. Calculate the autocovariance given by

$$R_x(k, l) = E[x(m, n)x(m + k, n + l)]$$

and its associated power spectral density $S_x(e^{j\mu}, e^{j\nu})$.

e) (10pt) Calculate $S_y(e^{j\mu}, e^{j\nu})$, the power spectral density of $y(m, n)$.