1. (20 pts) Find the ellipse
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \]
that comes as close as possible to the three data points:
\[ (x_1, y_1) = (1, 0), \quad (x_2, y_2) = (0, \sqrt{2}), \quad (x_3, y_3) = (1, 1). \]

2. (20 pts) Use the simplex method to solve the following linear program,

\[
\begin{align*}
\text{maximize} & \quad x_1 + 2x_2 \\
\text{subject to} & \quad -2x_1 + x_2 \leq 2 \\
& \quad x_1 - x_2 \geq -3 \\
& \quad x_1 \leq 3 \\
& \quad x_1 \geq 0, \ x_2 \geq 0.
\end{align*}
\]

3. (20 pts) Consider the following model of a linear, discrete, time-invariant system,
\[ x_{k+1} = Ax_k + Bu_k, \quad 0 \leq k \leq N - 1, \]
with a specified initial condition \( x_0 \) and a specified final state \( x_N = x_f \), where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), and \( N \geq n \). We assume that the pair \((A, B)\) is reachable. Use the Lagrange multiplier approach to calculate the optimal control sequence
\[ \{u_0, u_1, \ldots, u_{N-1}\} \]
that transfers \( x_0 \) to \( x_f \) while minimizing the quadratic performance index
\[ J_N = \frac{1}{2} \sum_{k=0}^{N-1} u_k^T R u_k, \]

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where \( R = R^T > 0 \).

**Hint**: Define the composite input vector

\[
    u = \begin{bmatrix} u_0^T & u_1^T & \cdots & u_{N-1}^T \end{bmatrix}^T
\]

and the symmetric block-diagonal positive-definite matrix

\[
    L = \begin{bmatrix}
        R & O & \cdots & O \\
        O & R & \cdots & O \\
        \vdots & \ddots & \ddots & \vdots \\
        O & O & \cdots & R
    \end{bmatrix}
\]

It is then easy to verify that the performance index \( J_N \) can be represented as

\[
    J_N = \frac{1}{2} u^T L u.
\]

Next, write the plant model in the form

\[
    M u = f
\]

for some matrix \( M \) and a vector \( f \).

(i) **(5 pts)** Give expressions for \( M \) and \( f \). Note that the expression for \( f \) is in terms of \( x_f \) and \( x_0 \).

(ii) **(5 pts)** Represent the problem of optimal transfer of the system from the initial state \( x_0 \) to the final state \( x_f \) as a constrained optimization problem.

(iii) **(10 pts)** Obtain a closed-form expression for \( u \).

4. **(20 pts)** Consider a square matrix \( Q \) partitioned as follows:

\[
    Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},
\]

where \( Q_{11} \) and \( Q_{22} \) are square submatrices. If \( Q_{11} \) is nonsingular, then we can write

\[
    \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} I & O \\ Q_{21} & I \\ Q_{21} Q_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} Q_{11} & O \\ O & \Delta \end{bmatrix} \begin{bmatrix} I & O_{12} \\ O & I \end{bmatrix},
\]
where $\Delta = Q_{22} - Q_{21} Q_{11}^{-1} Q_{12}$ is called the Schur complement of $Q_{11}$. Suppose now that $Q$ is symmetric, that is, $Q = Q^T$.

(i) (10 pts.) Formulate necessary and sufficient conditions for $Q$ to be positive definite in terms of $Q_{11}$, $Q_{12}$, and $Q_{22}$;

(ii) (10 pts.) Assume that $Q_{22}$ is nonsingular. Find an expression for the Schur complement of $Q_{22}$.

5. (20 pts) Given a monotone non-decreasing function $g$ of single variable, that is, $g(r_1) \leq g(r_2)$ for $r_1 < r_2$. The function $g$ is also convex. Let $f$ be a convex function on a convex set $\Omega \subseteq \mathbb{R}^n$. Show that the composite function $g(f(x))$ is convex on $\Omega$. 

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