Continuous-Time Fourier Transform: \( X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \)

Continuous-Time Fourier Transform Pair: \( \mathcal{F}\left\{ \frac{\sin(Wt)}{\pi t} \right\} = \text{rect}\left\{ \frac{\omega}{2W} \right\} \) where \( \text{rect}(x) = 1 \) for \( |x| < 0.5 \) and \( \text{rect}(x) = 0 \) for \( |x| > 0.5 \).

Continuous-Time Fourier Transform Property: \( \mathcal{F}\{x_1(t)x_2(t)\} = \frac{1}{2\pi} X_1(\omega) \ast X_2(\omega) \), where * denotes convolution, and \( \mathcal{F}\{x_i(t)\} = X_i(\omega), i = 1, 2 \).

Discrete-Time Fourier Transform: \( X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \)

**Problem 1.** [42 pts]

**Problem 1 (a).** Consider an analog signal with maximum frequency (bandwidth) \( \omega_M = 20 \) rads/sec. That is, the Fourier Transform of the analog signal \( x_a(t) \) is exactly zero for \( |\omega| > 20 \) rads/sec. This signal is sampled at a rate \( \omega_s = 60 \) rads/sec, where \( \omega_s = 2\pi/T_s \) such that the time between samples is \( T_s = \frac{2\pi}{60} \) sec. This yields the discrete-time sequence

\[
x[n] = x_a(nT_s) = \left( \frac{60}{2\pi} \right)^2 \frac{\sin(\frac{\pi}{6}n)}{\pi n} \frac{\sin(\frac{\pi}{2}n)}{\pi n} \quad \text{where: } \quad T_s = \frac{2\pi}{60}
\]

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal \( x_r(t) \). Show all work.

\[
x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s)
\]

where: \( T_s = \frac{2\pi}{60} \) and \( h(t) = T_s \frac{\pi}{10} \frac{\sin(10t)}{\pi t} \)

**Problem 1 (b).** Consider the SAME analog signal with maximum frequency (bandwidth) \( \omega_M = 20 \) rads/sec. This signal is sampled at the same rate \( \omega_s = 60 \) rads/sec., where \( \omega_s = 2\pi/T_s \) and the time between samples is \( T_s = \frac{2\pi}{60} \) sec, but at a different starting point. This yields the Discrete-Time \( x[n] \) signal below, where \( 0 < \epsilon < 1 \).

\[
x_c[n] = x_a(nT_s + \epsilon T_s) = \left( \frac{60}{2\pi} \right)^2 \frac{\sin(\frac{\pi}{6}(n + \epsilon))}{\pi (n + \epsilon)} \frac{\sin(\frac{\pi}{2}(n + \epsilon))}{\pi (n + \epsilon)} \quad \text{where: } \quad T_s = \frac{2\pi}{60}
\]

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal \( x_r(t) \). Does your final answer depend on the value of \( \epsilon \)? Explain your answer.

\[
x_r(t) = \sum_{n=-\infty}^{\infty} x_c[n]h(t - (n + \epsilon)T_s)
\]

where: \( T_s = \frac{2\pi}{60} \) and \( h(t) = T_s \frac{\pi}{10} \frac{\sin(10t)}{\pi t} \)
Problem 1 (c). Consider the SAME analog signal with maximum frequency (bandwidth) \( \omega_M = 20 \text{ rads/sec.} \) This signal is sampled at a rate \( \omega_s = 40 \text{ rads/sec., where } \omega_s = 2\pi/T_s \) such the time between samples is \( T_s = \frac{2\pi}{40} \text{ sec, yielding the following discrete-time sequence:} \)

\[
x[n] = x_a(nT_s) = \left( \frac{40}{2\pi} \right)^2 \frac{\sin(\frac{\pi}{4}n)}{\pi n} \frac{\sin(\frac{3\pi}{4}n)}{\pi n} \quad \text{where: } T_s = \frac{2\pi}{40}
\]

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal \( x_r(t) \). Show all work.

\[
x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \quad \text{where: } T_s = \frac{2\pi}{40} \text{ and } h(t) = T_s \frac{\sin(20t)}{\pi t}
\]

Problem 1 (d). Consider the SAME analog signal with maximum frequency (bandwidth) \( \omega_M = 20 \text{ rads/sec.} \) This signal is sampled at the same rate \( \omega_s = 40 \text{ rads/sec., where } \omega_s = 2\pi/T_s \) and the time between samples is \( T_s = \frac{2\pi}{40} \text{ sec, but at a different starting point.} \) This yields the Discrete-Time \( x[n] \) signal below, where \( 0 < \epsilon < 1. \)

\[
x_e[n] = x_a(nT_s + \epsilon T_s) = \left( \frac{40}{2\pi} \right)^2 \frac{\sin(\frac{\pi}{4}(n + \epsilon))}{\pi(n + \epsilon)} \frac{\sin(\frac{3\pi}{4}(n + \epsilon))}{\pi(n + \epsilon)} \quad \text{where: } T_s = \frac{2\pi}{40}
\]

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal \( x_r(t) \). does your final answer depend on the value of \( \epsilon \)? Explain your answer.

\[
x_r(t) = \sum_{n=-\infty}^{\infty} x_e[n]h(t - (n + \epsilon)T_s) \quad \text{where: } T_s = \frac{2\pi}{40} \text{ and } h(t) = T_s \frac{\sin(20t)}{\pi t}
\]
Problem 1 (e). Consider the SAME analog signal with maximum frequency (bandwidth) \( \omega_M = 20 \text{ rads/sec} \). This signal is sampled at a rate \( \omega_s = 30 \text{ rads/sec} \), where \( \omega_s = 2\pi / T_s \) such the time between samples is \( T_s = \frac{2\pi}{30} \text{ sec} \), yielding the following discrete-time sequence:

\[
x[n] = x_a(nT_s) = \left( \frac{30}{2\pi} \right)^2 \frac{\sin \left( \frac{\pi}{3} n \right) \sin(\pi n)}{\pi n}
\]

where: \( T_s = \frac{2\pi}{30} \)

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal \( x_r(t) \). Show all work.

\[
x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \quad \text{where: } T_s = \frac{2\pi}{30} \text{ and } h(t) = T_s \frac{\sin(15t)}{\pi t}
\]

Problem 1 (f). Consider the SAME analog signal with maximum frequency (bandwidth) \( \omega_M = 20 \text{ rads/sec} \). This signal is sampled at the same rate \( \omega_s = 30 \text{ rads/sec} \), where \( \omega_s = 2\pi / T_s \) and the time between samples is \( T_s = \frac{2\pi}{30} \text{ sec} \), but at a different starting point. This yields the Discrete-Time \( x[n] \) signal below, where \( 0 < \epsilon < 1 \).

\[
x_e[n] = x_a(nT_s + \epsilon T_s) = \left( \frac{30}{2\pi} \right)^2 \frac{\sin \left( \frac{\pi}{3} (n + \epsilon) \right) \sin(\pi (n + \epsilon))}{\pi (n + \epsilon)}
\]

where: \( T_s = \frac{2\pi}{30} \)

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal \( x_r(t) \). does your final answer depend on the value of \( \epsilon \)? Explain your answer.

\[
x_r(t) = \sum_{n=-\infty}^{\infty} x_e[n] h(t - (n + \epsilon)T_s) \quad \text{where: } T_s = \frac{2\pi}{30} \text{ and } h(t) = T_s \frac{\sin(15t)}{\pi t}
\]

Problem 2. [20 pts] The decaying exponential signal \( x_a(t) = e^{-ln(2)t} u(t) \) is sampled every \( T_s = 1 \text{ second} \) to form \( x[n] = x_a(nT_s) \). The Fourier Transform of \( x_a(t) \) is \( X_a(\omega) \) is not strictly band-limited so there will always be some amount of aliasing. We know that the DTFT of \( x[n] \) is related to the CTFT \( X_a(\omega) \) according to the expressions below, where \( F_s = 1 \) and \( \omega_s = 2\pi \), since \( T_s = 1 \text{ sec} \):

\[
X(\omega) = X_s(F_s\omega) \quad \text{where: } X_s(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)
\]

Determine a closed-form expression for the DTFT \( X(\omega) \). Show all work.
Problem 3. [38 pts]

(a) **Note:** System 1 and System 2 defined in parts (b) and (c), respectively, are NOT in parallel, but they do have the same input equal to \( x[n] \) below with \( p = \frac{3}{4} \).

\[
x[n] = \frac{1}{p} \left\{ \delta[n] + (p^2 - 1)p^n u[n] \right\}
\]  

(1)

Determine and plot the autocorrelation sequence \( r_{xx}[\ell] \) for \( x[n] \) defined above with \( p = \frac{3}{4} \). The autocorrelation for \( x[n] \) is defined as \( r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]x^*[n-\ell] \).

(b) Consider System 1 below:

System 1: \( y_1[n] = x[n] + x[n-1] - x[n-2] + x[n-3] \)

\[ x[n] \rightarrow \begin{array}{c} h_1[n] \end{array} \rightarrow y_1[n] \]

(i) Determine the impulse response \( h_1[n] \). You can write it in sequence form.

(ii) Determine the autocorrelation of the impulse response. Do a stem plot of \( r_{h_1 h_1}[\ell] \).

(iii) For the input signal in Equation (1) above with \( p = \frac{3}{4} \), determine the autocorrelation of the output \( y_1[n] \). Do a stem plot of \( r_{y_1 y_1}[\ell] \).

(c) Consider System 2 below:


\[ x[n] \rightarrow \begin{array}{c} h_2[n] \end{array} \rightarrow y_2[n] \]

(i) Determine the impulse response \( h_2[n] \). You can write it in sequence form.

(ii) Determine the autocorrelation of the impulse response. Do a stem plot of \( r_{h_2 h_2}[\ell] \).

(iii) Determine the autocorrelation of the output \( y_2[n] \). Do a stem plot of \( r_{y_2 y_2}[\ell] \).

(d) Using your answers from parts (b) and (c), do a stem plot of \( r[\ell] = r_{y_1 y_1}[\ell] + r_{y_2 y_2}[\ell] \).