Q1 (20 points).

(a) Assume the run time of some algorithm is given by the following recurrence:

\[ T(n) = 2T(\sqrt{n}) + \log n. \]

Find the asymptotic run time complexity of this algorithm. Give detail of your computation.

(b) Assume functions \( f \) and \( g \) such that \( f(n) \) is \( O(g(n)) \). Prove or disprove that \( 3^{f(n)} \) is \( O(3^{g(n)}) \).

Q2 (30 points): Suppose your company develops and manages construction of boat launching docks along a downstream stretch of Wabash river. This stretch runs north-south for \( L \) miles within the State of Indiana. The possible sites for docks are given by numbers \( x_1 < x_2 < x_3 < \ldots < x_n \), each in the interval \([0, L]\), specifying their position in miles measured from the northern end of this stretch of Wabash. If your company constructs a dock at position \( x_i \), it receives a revenue of \( r_i > 0 \). Regulations imposed by the Indiana Department of Water Resource Management require that no two docks should be built within a distance of 5 miles or less from each other. Your company plans to construct docks at a subset of the potential sites so as to maximize the total revenue, subject to this distance restriction. For example, suppose \( L = 20 \) and \( n = 5 \) with potential sites given by \( \{x_1, x_2, x_3, x_4, x_5\} = \{6, 7, 12, 13, 14\} \) and \( \{r_1, r_2, r_3, r_4, r_5\} = \{5, 6, 5, 3, 1\} \). Then the best solution is to construct docks at locations \( x_1 \) and \( x_3 \) to achieve revenue of 10.

Describe a dynamic programming formulation to find a solution for this optimization problem. Compute the complexity of solving your dynamic programming formulation of this problem.
Q3 (25 points): The minimum bottleneck spanning tree (MBST) is a spanning tree that seeks to minimize the use of the most expensive (the largest weight) edge in the tree. More specifically, for a tree $T$ over a graph $G$, we define $e$ to be a bottleneck edge of $T$ if it's an edge with the largest weight. Note, multiple edges may have the same weight. The tree $T$ is an MBST if it is a spanning tree and there is no other spanning tree of $G$ with a cheaper bottleneck edge. Prove or disprove that an MBST for a graph $G$ is always a minimum spanning tree for $G$.

Q4 (25 points). A project management firm needs to hire technical experts for a project which requires $s$ different specialties. The project requires at least one expert in each of the specialties. The firm has received job applications from $t$ potential individuals. An applicant may have multiple expertises. For each of the $s$ specialties, there is some subset of the $t$ applicants qualified in the required technical areas. For a given number $k < t$, is it possible to hire at most $k$ applicants and have at least one expert qualified in each of the $s$ specialties? Prove or disprove that this problem is NP-complete.