Problem 1. (50 points)

An insulating sphere with radius $a$ is immersed in a conducting medium with conductivity $\sigma$ and relative dielectric constant $\varepsilon_r$. There is a uniform background electric field in the conductor and far from the sphere the electric field is $E=E_0 \hat{z}$.

Let the origin of the sphere coordinate system be at the center of the sphere. Assume that the electric field is DC - i.e. it is steady in time.

(a) (45 points) Derive an expression for the current density in the region outside the sphere.

(b) (5 points). What is the current density inside the sphere? Explain your answer.

Possibly useful information.

With $\phi$ symmetry, Laplace's equation in spherical coordinates is

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) = 0 \quad (1)$$

The solution to Eq. 1 is

$$\Phi(r, \theta) = \sum_{n=0}^{\infty} \left( A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta) \quad (2)$$

$P_n$ are Legendre polynomials. The first few of which are

$$P_0 = 1 \quad P_1 = \cos \theta \quad P_2 = \frac{1}{2} \left( 3 \cos^2 \theta - 1 \right)$$

$$\nabla \Phi = \frac{\partial \Phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \hat{\theta} \quad (3)$$
Problem 2.  (50 points)

A solenoidal coil is filled with water with NaCl added to yield a conductivity $\sigma$ (units S/m). The height of the solenoid is $h$, radius is $a$, and there are $N$ turns. Assume $h \gg a$. The winding is connected to a continuous wave electric power source with angular frequency $\omega$, voltage amplitude $V_0$ and current amplitude $I_0$.

Assume that the conductivity and frequency are sufficiently low so that the magnetic field produced by the eddy currents in the salt water is small compared to the magnetic field produced by the coil.

\[ V_0, I_0 \]

(a) (20 points). Derive an expression for the magnetic field inside the solenoid.

(b) (20 points). Derive an expression for the average power dissipation in the salt water inside the solenoid.

(c) (10 points). What is the impedance $Z=V_0/I_0$ of the solenoid?