I. Fourier-Based Imaging (60 points)

For the following, consider an image \( f(x, y) \) with a (continuous two-dimensional) Fourier transform given by \( F(u, v) \) and a forward projection

\[
p_b(r) = \mathcal{F} \mathcal{P} \{ f(x, y) \} \\
= \int_{-\infty}^{\infty} f(r \cos(\theta) - z \sin(\theta), r \sin(\theta) + z \cos(\theta)) \, dz
\]

and let \( P_b(\rho) \) denote the (continuous time) Fourier transform of \( p_b(r) \).

Define the functions

\[
\delta(x, y) = \delta(x) \delta(y) \\
\text{rect}(x) = \begin{cases} 
1 & \text{if } |x| \leq 1/2 \\
0 & \text{if } |x| > 1/2
\end{cases}
\]

Sampling of the signal \( f(x, y) \) may be (ideally) represented as multiplication by the \textit{comb} function:

\[
\delta_s(x, y; \Delta x, \Delta y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)
\]

In this case, the sampled signal is

\[
f_s(x, y) = f(x, y) \times \delta_s(x, y; \Delta x, \Delta y)
\]

(20 pts) 1. Symbolically derive \( F_s(u, v) \), the 2D Fourier transform of \( f_s(x, y) \). (Show all intermediate steps.)

(10 pts) 2. For \( f(x, y) \) frequency band-limited to \(|u| < U \) and \(|v| < V \), derive and graphically illustrate the Nyquist sampling criteria related to \( \Delta x \) and \( \Delta y \).

(10 pts) 3. Assume \( f(x, y) \) is spatially limited to \(|x| < \frac{X}{2} \) and \(|y| < \frac{Y}{2} \). If a \( X \times Y \) image of \( f(x, y) \) is generated using \( n_x \times n_y \) pixels, what is the set of points in \((u, v)\) space that have been sampled?

(10 pts) 4. If the "field-of-view" in part c is doubled, to produce a \( 2X \times 2Y \) image, without increasing the number of pixels, how does the set of sampled points in \((u, v)\) space change?

(10 pts) 5. Now assume that the "field-of-view" remains \( X \times Y \), but the number of pixels is doubled to \( 2n_x \times 2n_y \). How does the set of sampled points in \((u, v)\) space change?
II. Projection/Emission Tomographic Imaging (40 points)

(20 pts) 6. Assume that you have received projection data from a tomographic imaging system. In this case, an array of detectors lies along a line that is tangential to the radius of the imaged object. Each detector results in a single recorded value per measurement, with this value being proportional to the accumulated effect of a property (either absorption or emission) that varies along a line through the imaged object — i.e., the value represents a line integral. The “lines of response” as measured across the detector array may be reformulated into a sinogram, which is then used to reconstruct the distribution of the given property within the target by one of a number of methodologies. One of the conceptually simplest techniques is the algebraic reconstruction technique (ART). Illustrate the ART reconstruction procedure associated with two projection angles obtained from a $3 \times 3$ image, given the following sinogram components:

\[
\begin{align*}
p_{\theta=0}(r) &= \begin{bmatrix} 11 & 14 & 8 \end{bmatrix} \\
p_{\theta}(r) &= \begin{bmatrix} 14 & 10 & 9 \end{bmatrix}
\end{align*}
\]

Note: Assume that $\theta = 0$ implies a projection line orthogonal to the x-axis.

(20 pts) 7. Suppose that for an emission tomographic system, detector units are tightly packed in a ring around the to-be-imaged object. Assume in this case that each detector unit comprises four square photomultiplier tubes (PMTs) ($2 \times 2$ matrix) fronted by a single scintillation crystal with slits made in such a way that it is divided into an $8 \times 8$ matrix of individual detectors. Note: Assume that the center of the $(i, j)^{th}$ PMT is at $(x_i, y_j)$, and that the PMTs and the detectors cover the exact same square area.

In this case, we will assume that the response of a PMT to an event occurring in a particular subcrystal may be modeled as

\[ a_{PMT} = e^{-\tau} \]

where $r$ is the distance from the center of the PMT to the center of the subcrystal, and $\tau$ is the spatial length constant of propagation of the light in the crystal.

Find a general expression for the response in the $(i, j)^{th}$ PMT to an event in the $(k, l)^{th}$ subcrystal. Remember to provide a diagram to indicate the numeric ordering of your PMTs and subcrystals.