LTI and LT Systems – State-Space Approach

Unless otherwise stated, you need to justify your answers.

Problem 1. (45 points) Consider the continuous-time linear time-invariant system

\[ x(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) , \]

where \( x(t) \in \mathbb{R}^3, u(t), y(t) \in \mathbb{R}, \) and matrices \( A, B, C \) are given by

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & -1 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
1 \\
0 \\
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 1 & 1 \\
\end{bmatrix}. 
\]

(a) (5 pts) Is the system controllable?

(b) (5 pts) What is the controllable subspace of the system?

(c) (5 pts) Is the system observable?

(d) (5 pts) What is the unobservable subspace of the system?

(e) (5 pts) Does there exist a state feedback gain \( K \in \mathbb{R}^{1 \times 3} \) such that \( A - BK \) has eigenvalues \([-2, -2, -2]\)? If the answer is yes, find such a \( K \). Otherwise, state your reason.

(f) (5 pts) Does there exist an \( L \in \mathbb{R}^{3 \times 1} \) such that \( A - LC \) has eigenvalues \( \{1, -1, -2\} \)? If the answer is yes, find such an \( L \). Otherwise, state your reason.

(g) (5 pts) Assume \( u(t) \equiv 0 \). Is the resulting autonomous system stable?

(h) (5 pts) Assume \( u(t) \equiv 0 \). Is it true that, starting from any \( x(0), y(t) \) will be bounded, i.e., \( |y(t)| \leq K \) for all \( t \geq 0 \) for some constant \( K \) (possibly dependent on \( x(0) \)?)

(i) (5 pts) Is the system BIBO stable? In other words, assuming \( x(0) = 0 \), is there a constant \( K \) so that \( |u(t)| \leq 1, \forall t \geq 0 \), implies \( |y(t)| \leq K, \forall t \geq 0 \)?

Problem 2. (20 points) The following discrete-time LTI system is given:

\[ x[k+1] = Ax[k] + Bu[k] = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k], \quad k = 0, 1, \ldots \]

(a) (10 pts) Compute \( A^k \) for all \( k = 0, 1, \ldots \) and write the expression of \( x[k] \) starting from the initial condition \( x[0] \) and under arbitrary inputs \( u[k], k = 0, 1, \ldots \).

(b) (10 pts) Find the control \( u[0], u[1], u[2] \) with the least energy \((u[0])^2 + (u[1])^2 + (u[2])^2\) that can steer the system from the initial state \( x[0] = [0 \ 0]^T \) to \( x[3] = [1 \ 1]^T \) at time \( k = 3 \).
Problem 3. (20 points) Find the state transition matrix \( \Phi(t, s) \), \( t \geq s \geq 0 \), of the following linear time-varying system:

\[
\dot{x}(t) = A(t)x(t) = \begin{bmatrix} -2t & t \\ 0 & -t \end{bmatrix} x(t).
\]

Problem 4. (15 points) Consider the following nonlinear system

\[
\begin{align*}
\dot{x}_1 &= x_1(2 - x_2) \\
\dot{x}_2 &= -x_2(1 - x_1).
\end{align*}
\]

The system has two equilibrium points \( x_{e,1} = [0, 0]^T \) and \( x_{e,2} = [1, 2]^T \).

(a) (10 pts) Construct the linearized state models around the above two equilibrium points, respectively.

(b) (5 pts) Discuss the local stability of the nonlinear system around \( x_{e,1} \) and \( x_{e,2} \) based on the linearized models. What conclusions can you draw?