Two electromagnetic plane waves propagate through vacuum in directions defined by \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) as shown, where \( \mathbf{k}_i = \frac{2\pi}{\lambda} (-\sin \theta_i \hat{x} + \cos \theta_i \hat{z}) \) for \( i = 1, 2 \). For each, the electric wave can be expressed as \( \mathbf{E}_i = \mathbf{E}_{i,0} e^{-j(\mathbf{k}_i \cdot \mathbf{r} - \omega t)} \).

A) **(26 points)** For \( \omega_1 = \omega_2 \), determine the distance \( \Lambda \) between the time-averaged power density maxima in the plane defined by \( z = 0 \). Express your result in terms of \( \lambda = \frac{2\pi}{|\mathbf{k}|}, \theta_1 \) and \( \theta_2 \). Determine \( \Lambda \) for \( \lambda = 1.0 \ \mu m, \theta_1 = 0.01 \ \text{rad} \) and \( \theta_2 = -0.01 \ \text{rad} \).

B) **(27 points)** Now let \( \omega_1 \) and \( \omega_2 \) be similar to, but slightly different from one another (i.e., \( |\omega_1 - \omega_2| \ll \omega_1, \omega_2 \)), and let \( \theta_1 = -\theta_2 \). Determine the velocity \( v \) of the time-averaged power density peaks in the \( z = 0 \) plane in terms of \( \lambda, \theta_1, \theta_2 \), and \( \Delta \omega = \omega_1 - \omega_2 \). Estimate \( v \) for \( \lambda = 1.0 \ \mu m, \theta_1 = -\theta_2 = 0.01 \ \text{rad} \) and \( \Delta \omega = 2\pi \times 10^7 \ \text{rad}/\text{sec} \).

C) **(27 points)** Consider the two waves of frequency \( \omega_1 \) and \( \omega_2 \) (again with \( |\omega_1 - \omega_2| \ll \omega_1, \omega_2 \)) as they propagate collinearly (\( \theta_1 = \theta_2 = 0 \)) through a dispersive, non-absorbing, isotropic, non-magnetic medium. The relative permittivity of the medium is given by

\[
\varepsilon'_r(\omega) = \varepsilon'_r(\omega_0) + \varepsilon''_r(\omega_0)(\omega - \omega_0) + \ldots,
\]

where \( \varepsilon'_r(\omega_0) = \left. \frac{d\varepsilon'_r(\omega)}{d\omega} \right|_{\omega=\omega_0} \) and \( \omega_0 \) is the average frequency given by \( \omega_0 = \frac{(\omega_1 + \omega_2)}{2} \). Derive an expression for the velocity of the time-averaged power density peak of this waveform, travelling in the +z direction. Your answer should be in terms of \( \omega_0, c = (\varepsilon_0 \mu_0)^{-1/2}, \varepsilon'_r(\omega_0) \) and \( \varepsilon''_r(\omega_0) \). Estimate this velocity for \( \lambda_0 = 1.0 \ \mu m \) (wavelength in free space), \( \varepsilon'_r(\omega_0) = 2, \varepsilon''_r(\omega_0) = \frac{10^{-11}}{2\pi} \ \text{rad/sec}^{-1} \) and \( \Delta \omega = 2\pi \times 10^7 \ \text{rad/sec} \).
D) (20 points) Finally, we consider a large number \( N \) of plane waves, each propagating through vacuum (impedance \( \eta \approx 377 \, \Omega \)) at an angle \( \theta_n = n\Delta \theta \) with respect to the \( z \) axis, where \( n \) is an integer between \( -\frac{N}{2} \) and \( +\frac{N}{2} \). Let \( N\Delta \theta \ll 1 \).

\[
E_n = E_0 e^{-j(k_n \hat{r} - \omega_0 t)}
\]

\[
k_n = \frac{2\pi}{\lambda} (-\sin \theta_n \hat{x} + \cos \theta_n \hat{z}).
\]

The amplitude \( E_0 \) and frequency \( \omega_0 \) are the same for all waves.

a) What is the peak time-averaged power density in the \( z = 0 \) plane in terms of \( N \), \( E_0 \) and \( \eta \)?

b) Show that the distance from the axis to the first zero of the time-averaged power density is \( x_o = \frac{\lambda}{N\Delta \theta} \).

c) Find the distance \( x_m \) from the axis to the next time-averaged power density maximum.

d) What is the time-averaged power density of the peak at \( x = x_m \) relative to that of the central peak?

[Hint: Represent the sum of these waves using a phasor diagram. The magnitude of each individual phasor is \( E_0 \) and the phase difference between phasors is \( \delta \). Express \( \delta \) as a function of \( x \). As \( N \) gets large, the phasor sum representing the total field is the arc of a circle. Draw this phasor diagram for the field at the central peak. Repeat for the field at the first zero, and the next maximum.]

Write in Exam Book Only
Maxwell’s Equations:
\[ \nabla \cdot D = \rho_v \]
\[ \nabla \cdot B = 0 \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]
\[ \nabla \times H = J + \frac{\partial D}{\partial t} \]
\[ \oint_S D \cdot dS = Q_{enc} \]
\[ \oint_S B \cdot dS = 0 \]
\[ \oint_c E \cdot dl = -\frac{d}{dt} \oint_S B \cdot dS \]
\[ \oint_c H \cdot dl = I_{enc} + \frac{d}{dt} \oint_S D \cdot dS \]

Poynting’s Theorem:
\[ \nabla \cdot (E \times H) = -\frac{\partial}{\partial t} (B \cdot H) - \frac{\partial}{\partial t} (D \cdot E) - J \cdot E \]

Potentially Useful Vector Algebra
\[ \nabla \cdot A = \hat{x} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{y} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{z} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \]
\[ \nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \]
\[ \nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \]

Potentially Useful Integral Identities
\[ \int \frac{dx}{x} = \ln x \]
\[ \int \frac{dx}{\sqrt{x^2 + a^2}} = \ln \left( x + \sqrt{x^2 + a^2} \right) \]
\[ \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \]
\[ \int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2 \sqrt{x^2 + a^2}} \]
\[ \int \sin^2 x dx = x - \frac{1}{4} \sin 2x \]
\[ \int \sin^3 x dx = \frac{\cos^3 x}{3} - \cos x \]
\[ \int \sinh^2 x dx = \frac{1}{2} [-x + \sinh x \cosh x] \]
\[ \int \cosh^2 x dx = \frac{1}{2} [x + \sinh x \cosh x] \]

Other Information
\[ \nu_g = \frac{d\omega}{dk}, \quad k = \omega \sqrt{\mu \varepsilon} \]