Q1 (20 points). Assume functions \( f \) and \( g \) such that \( f(n) = O(g(n)) \). For each of the following statements, decide whether it is true or false and give a proof or counterexample.

(a) \( 2^{f(n)} = O(2^{g(n)}) \).

(b) \( f(n)^2 = O(g(n)^2) \).

Q2 (30 points): Let \( G \) be a graph of \( n \) nodes connected in the form of a path with weights attached to its nodes. A subset of the nodes is called an independent set if no two of them are joined by an edge. Give an algorithm that takes this \( n \)-node path graph with weights and returns an independent set of maximum total weight (an example of a 5-node path graph with weights and with the resulting maximal weight of the independent set is shown below). The running time of your algorithm should be polynomial in \( n \) and independent of the values of the weights.

\[
\begin{array}{c}
\text{1} & \text{8} & \text{6} & \text{3} & \text{6}
\end{array}
\]

Note: The maximum weight of an independent set of this example is 14.

Q3 (25 points): Given a graph \( G \) and a minimum spanning tree \( T \), suppose that we decrease the weight of one of the edges in \( T \). Show that \( T \) is still a minimum spanning tree for \( G \). More formally, let \( T \) be a minimum spanning tree for \( G \) with edge weights given by weight function \( w \). Choose one edge \( (x, y) \in T \) and a positive number \( k \), and define the weight function \( w' \) by

\[
w'(u, v) = \begin{cases} 
w(u, v) & \text{if } (u, v) \neq (x, y), \\
w(x, y) - k & \text{if } (u, v) = (x, y). \end{cases}
\]

Show that \( T \) is a minimum spanning tree for \( G \) with edge weights given by \( w' \).

Q4 (25 points). Suppose you’re helping to organize a summer sports camp, and the following problem comes up. The camp is supposed to have at least one counselor who’s skilled at each of the \( n \) sports covered by the camp (baseball, volleyball, and so on). They have received job applications from \( m \) potential counselors. For each of the \( n \) sports, there is some subset of the \( m \) applicants qualified in the sport. The question is: For a given number \( k < m \), is it possible to hire at most \( k \) of the counselors and have at least one counselor qualified in each of the \( n \) sports? We’ll call this problem the Efficient Recruiting Problem.

Show that Efficient Recruiting is NP-complete.