1. (20 points)
Consider an electromagnetic field in a source-free region in vacuum with electric field vector given by
\[ \mathbf{E} = \hat{y} E_0 \cos\left(6\pi \cdot 10^9 t - \beta z\right) \]
where the units of \( E, t, \) and \( z \) are volts/meter, seconds, and meters, respectively (all quantities are in MKS units).
  a) Find the magnetic field vector \( \mathbf{H} \).
  b) Give numerical values for \( \beta \) and for the wavelength, including units.

2. (30 points)
Consider an electromagnetic field in a source-free region in vacuum with electric field vector given by
\[ \mathbf{E} = \hat{y} E_0 \sin\left(10\pi x\right) \cos\left(6\pi \cdot 10^9 t - \beta z\right) \]
where all quantities are in MKS units.
  a) Find the magnetic field vector \( \mathbf{H} \).
  b) Give a numerical value for \( \beta \).
  c) We wish to position two perfectly conducting metal plates such that in the region between the plates, the electromagnetic fields are undisturbed. Carefully specify the position and orientation of the metal plates and explain clearly how boundary conditions are satisfied for each of Maxwell’s four equations. You may make use of formulas on the attached sheet if desired.

3. (25 points)
Poynting’s theorem for linear, time-invariant media can be written as follows:
\[ \int \mathbf{V} \left[ \frac{\partial}{\partial t} \left( \frac{\mu \mathbf{H} \cdot \mathbf{H}}{2} \right) + \frac{\partial}{\partial t} \left( \frac{\varepsilon \mathbf{E} \cdot \mathbf{E}}{2} \right) + \mathbf{E} \cdot \mathbf{J} \right] dV = -\oint \mathbf{S} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} \]
Explain in words the physical meaning of Poynting’s theorem, individually discussing each of the four terms in this equation. Give units! A description of 2-3 sentences per term should be sufficient.

4. (25 points)
We consider a lossless, air-filled, metallic resonator of cubic shape (all sides have length \( a \)). The fields for the \( \text{TE}_{10p} \) mode are given in phasor form as:
\[ \tilde{E}_y = E_0 \sin \frac{\pi x}{a} \sin \frac{p\pi z}{a} \quad \tilde{H}_x = -j \frac{E_0}{\eta} \frac{p\lambda}{a} \sin \frac{\pi x}{a} \cos \frac{p\pi z}{a} \quad \tilde{H}_z = j \frac{E_0}{\eta} \frac{\lambda}{2a} \cos \frac{\pi x}{a} \sin \frac{p\pi z}{a} \]
Give an expression for the free-space wavelength (\( \lambda \)) of the plane-waves making up this \( \text{TE}_{10p} \) resonant mode in terms of the resonator side \( a \) and the mode index \( p \). \( \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} \) is the impedance of free space.
Maxwell's equations (statics)

\[
\begin{array}{|c|c|c|}
\hline
\oint \mathbf{D} \cdot dS &= \int \rho_{\text{free}} dV & \nabla \cdot \mathbf{D} &= \rho_{\text{free}} & D_{12} - D_{2n} = \rho_s, \text{free} \\
\phi \mathbf{E} \cdot d\ell &= 0 & \nabla \times \mathbf{E} &= 0 & E_{H1} - E_{H2} = 0 \\
\oint \mathbf{B} \cdot dS &= 0 & \nabla \cdot \mathbf{B} &= 0 & B_{lin} - B_{2n} = 0 \\
\oint \mathbf{H} \cdot d\ell &= \int J_{\text{free}} \cdot dS & \nabla \times \mathbf{H} &= \mathbf{J}_{\text{free}} & H_{11} - H_{21} = J_s, \text{free} \\
\hline
\end{array}
\]

Maxwell's equations (time domain)

\[
\begin{align*}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
\end{align*}
\]

Vector Differential Operations

\[
\begin{align*}
\nabla \Phi &= \hat{x} \frac{\partial \Phi}{\partial x} + \hat{y} \frac{\partial \Phi}{\partial y} + \hat{z} \frac{\partial \Phi}{\partial z} \\
\nabla \cdot \mathbf{D} &= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \\
\nabla \times \mathbf{H} &= \hat{x} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) + \hat{y} \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) + \hat{z} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \\
\nabla^2 \mathbf{A} &= \hat{x} \nabla^2 A_x + \hat{y} \nabla^2 A_y + \hat{z} \nabla^2 A_z
\end{align*}
\]

Vector Formulas

\[
\begin{align*}
\nabla (\Phi \psi) &= \Phi \nabla \psi + \psi \nabla \Phi \\
\nabla \cdot (\psi \mathbf{A}) &= \mathbf{A} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{A} \\
\n\nabla \times (\Phi \mathbf{A}) &= \nabla \Phi \times \mathbf{A} + \Phi \nabla \times \mathbf{A} \\
\n\nabla \cdot \nabla \Phi &= \nabla^2 \Phi \\
\n\nabla \cdot \nabla \mathbf{A} &= 0 \\
\n\nabla \times \nabla \mathbf{A} &= \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}
\end{align*}
\]

\[
\begin{align*}
\int_S \nabla \times \mathbf{A} \cdot dS &= \oint \mathbf{A} \cdot d\mathbf{l} \\
\int_V \nabla \cdot \mathbf{A} \; dV &= \int_S \mathbf{A} \cdot dS
\end{align*}
\]

Physical Constants

- Vacuum permittivity: \( \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m} \)
- Vacuum permeability: \( \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \)
- Electron charge magnitude: \( e = 1.6022 \times 10^{-19} \text{ C} \)
- Speed of light in vacuum: \( c = 2.997925 \times 10^8 \text{ m/s} \)