The following questions are predicated on the assumption that you are working with a positron emission tomography (PET) scanner that has detectors packed tightly in a ring with radius $R$ m.

1) (40 pts) When positron emission occurs due to radioactive decay, annihilation of the positron leads to two photons being emitted from the object toward the ring of detectors located at radius $R$. A forward simulation of the detection of these emissions may be generated from knowledge of the location of the source — $(x_0, y_0)$, corresponding to $(r_0, \theta_0)$ — and the (ideal) angles at which the two photons are emitted — $\theta$ and $\pi + \theta$. Based on this simple forward model, derive equations for the angles associated with the two points— $(R, \phi_1)$ and $(R, \phi_2)$— at which the photons will intersect the detector ring. Note: we are using two coordinate systems, one rectangular and one polar, each having its origin at the center of the detector ring.

2) (20 pts) During formation of an image by emission tomography, it is desirable that the target lie primarily within a small radius of the center of the detector ring. Explain why resolution of a fixed ring emission tomography scanner becomes worse as the location of the object becomes more distant from the center of the ring.

3) (10 pts) Suppose the detector in the ring comprises four square photomultiplier tubes (PMTs) (2 x 2 matrix), and a single scintillation crystal with slits made in such a way that it is divided into an 8 x 8 matrix of individual detectors. Note: Assume that the center of the $(i, j)^{th}$ PMT is at $(x_i, y_j)$, and that the PMTs and the detectors cover the exact same square area.

In this case, we will assume that the response of a PMT to an event occurring in a particular subcrystal may be modeled as

$$ a_{\text{PMT}} = e^{-\frac{r}{\tau}} $$

where $r$ is the distance from the center of the PMT to the center of the subcrystal, and $\tau$ is the spatial length constant of propagation of the light in the crystal.

Find a general expression for the response in the $(i, j)^{th}$ PMT to an event in the $(k, l)^{th}$ subcrystal. Remember to provide a diagram to indicate the numeric ordering of your PMTs and subcrystals.

4) (10 pts) The desired final image is the spatial distribution of the radioactive sources within the target. The first step in the localization of these sources is the identification of the pair of photon detections that is likely to have arisen from each decay event. Identification of a pairs of detections is achieved using a coincidence “event window” that is held constant.
for the imaging system. Explain and provide a mathematical basis for a (reasonable) nominal coincidence window for the given system.

5) (20 pts) The number of coincidence detections between a pair of detectors is proportional to the number of decay events that occurred along the line between them, and serves a proxy for the total quantity (i.e., the integral) of radioactive source along this line. Therefore, the "lines of response" between all detector pairs may be reformatted into a sinogram, which is then used to reconstruct the distribution of sources within the target by one of a number of methodologies. One of the conceptually simplest techniques is the algebraic reconstruction technique (ART). Illustrate one full iteration of ART reconstruction for a $3 \times 3$ image with the following sinogram components:

$$p_{0 \cdot 0}(r) = \begin{bmatrix} 12 & 12 & 12 \end{bmatrix}$$
$$p_{\frac{1}{4}}(r) = \begin{bmatrix} 7 & 19 & 1 \end{bmatrix}$$
$$p_{\frac{3}{4}}(r) = \begin{bmatrix} 12 & 12 & 12 \end{bmatrix}$$
$$p_{\frac{7}{4}}(r) = \begin{bmatrix} 5 & 17 & 3 \end{bmatrix}$$

*Note: Assume that $\theta = 0$ implies a projection line orthogonal to the x-axis.*