Convolutional Neural Networks

- Convolution with Vector Input and Output
- Single and Multi-Layer CNNs
- Upsampling Pooling and Stride
Limitations of Dense Neural Networks

- Too many parameters:
  - Number of parameters scale roughly as $N^2$ for $N$-dimensional inputs.
  - Processing a $1024 \times 1024$ image, requires $\sim 10^{12}$ parameters per layer.
  - Both training and inference is slow.

- The solution to many problems should be space-invariant:
  - Object recognition: Recognition shouldn’t depend on object position in image.
  - Image denoising: Noise removal should depend on content, not absolute location in the image.
Don’t be a Jumanji NPC

- How did you set up the optimization for your CNN?
  - I used Batch Normalization

- Oh OK, but what did you actually do? Like how did you do the normalization.
  - I used Batch Normalization

- Yeah, I know you used batch normalization, but how did you store the normalization parameters for inference, and what values did you use?
  - I used Batch Normalization

- Can you say anything other than, “I used batch normalization”
  - I used Batch Normalization

*Warning: I don’t understand popular culture, and I didn’t check with my kids before writing this.*
What is Convolution?

- Discrete-time convolution is defined as 

\[ x(j) = z(j) * w(j) = \sum_{k=-\infty}^{\infty} z(j - k) w(k) \]

\[ = w(j) * z(j) = \sum_{k=-\infty}^{\infty} w(j - k) z(k) \]

In 2D, convolution is given by 

\[ x(j_1, j_2) = w(j_1, j_2) * z(j_1, j_2) = z(j_1, j_2) * w(j_1, j_2) \]

\[ = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} z(j_1 - k_1, j_2 - k_2) w(k_1, k_2) \]

*CNN sometimes use a non-standard definition of convolution, but you can just time reverse \( w \) to convert. However, the non-standard definition leads to issues. So, we will adopt the standard definition.
Convolution with Finite Kernels

- Discrete-time convolution is defined as

\[ x(j) = w(j) * z(j) = \sum_{k=-p}^{p} z(j - k) w(k) \]

where \( w \) is of length \( 2p + 1 \)

- In 2D, convolution is given by

\[ x(j_1, j_2) = \sum_{k_1=-p}^{p} \sum_{k_2=-p}^{p} z(j_1 - k_1, j_2 - k_2) w(k_1, k_2) \]

where \( w \) is of size \( p \times p \)
2D Convolution with “Same” boundary

- For 3×3 filter is given by

\[ x(j_1, j_2) = \sum_{k_1=-1}^{1} \sum_{k_2=-1}^{1} z(j_1 - k_1, j_2 - k_2) w(k_1, k_2) \]

Tensor notation

\[ x_{j_1, j_2} = w_{(j_1, j_2)} * z_{(j_1, j_2)} \]

Assumes \( p \) is odd

Parenthesizes denote convolution

(0,0) point
For $3 \times 3$ filter is given by

$$x(j_1, j_2) = \sum_{k_1=-1}^{1} \sum_{k_2=-1}^{1} z(j_1 + 1 - k_1, j_2 + 1 - k_2) w(k_1, k_2)$$

Tensor notation

$$x^{j_1,j_2} = w_{(j_1,j_2)} * z^{(j_1,j_2)}$$

(0,0) points
2D Convolution with Vector Input

- Vector 2D convolution of depth \(d_i = 2\) and \(d_o = 1\)

\[
x(j_1, j_2) = \sum_{i=0}^{d-1} \sum_{k_1=p}^{p} \sum_{k_2=p}^{p} z(j_1 - k_1, j_2 - k_2, i) w(k_1, k_2, i)
\]

Tensor notation

\[
x^{j_1, j_2} = w_{(j_1, j_2),i} * z^{(j_1, j_2),i}
\]
2D Convolution with Vector Input/Output

- Vector 2D convolution of depth $d_i = 2$ and $d_o = 2$

$$x^{j_1,j_2,j_3} = w_{(j_1,j_2),i}^{j_3} * z_{(j_1,j_2),i}$$

Convolution Diagram

- **Sum notation**

\[ x(j_1, j_2, j_3) = \sum_{i=0}^{d_i-1} \sum_{k_1=-p}^{p} \sum_{k_2=-p}^{p} z(j_1 - k_1, j_2 - k_2, i) w(k_1, k_2, i, j_3) \]

- **Tensor notation**

\[ x_{j_1, j_2, j_3} = w_{(j_1, j_2), i} j_3 \ast z_{(j_1, j_2), i} \]

where

\[ w \text{ is } 5 \times 5 \times 16 \times 16 = 6,400 \]
\[ y \text{ is } 512 \times 512 \times 16 = 4,194,304 \]
\[ x \text{ is } 512 \times 512 \times 16 = 4,194,304 \]
Single Layer CNN

- Single layer 2D CNN example:

\[ x = f_\theta(z) = \sigma(w \ast z + b) \]

\[ x^{j_1,j_2,j_3} = \sigma(w_{(j_1,j_2),i}^{j_3} \ast z^{(j_1,j_2),i} + b^{j_3}) \]

- Tensor structures

\[ w \text{ is } 5 \times 5 \times 16 \times 16 = 6,400 \]
\[ b \text{ is } 16 = 16 \]

\[ z \text{ is } 512 \times 512 \times 16 = 4,194,304 \]
\[ x \text{ is } 512 \times 512 \times 16 = 4,194,304 \]
5 Layer CNN

- **Parameters:**
  - Layer 1: filter $5 \times 5 \times 1 \times 16$; offset $16$; ReLU; #params $416$
  - Layer 2: filter $5 \times 5 \times 16 \times 16$; offset $16$; ReLU; #params $6,416$
  - Layer 3: filter $5 \times 5 \times 16 \times 16$; offset $16$; ReLU; #params $6,416$
  - Layer 4: filter $5 \times 5 \times 16 \times 16$; offset $16$; ReLU; #params $6,416$
  - Layer 5: filter $5 \times 5 \times 16 \times 1$; offset $16$; ReLU; #params $416$
  - Total # params $= 20,080$
Simpler Diagram for 5 Layer CNN

- **Good news:**
  - Dramatic reduction in number of parameters as compare to dense network
  - Spatially invariant behavior (except for boundaries)
  - For interior layers, each pixel is formed by a 16 dimensional feature vector
  - Intuition: CNN can design its own features!

- **Bad news:**
  - Not very useful
  - ReLUs are fast but can be difficult to train
  - Spatial resolution of each layer is the same
  - Output isn’t appropriate for classification
**CNN Stride**

- Stride of \((d_v, d_h) = (2,2)\), results in

\[
x(j_1, j_2) = z(2j_1, 2j_2)
\]

- Comments:
  - Output has reduced dimensions of \(\left[\frac{N_v}{d_v}\right], \left[\frac{N_h}{d_h}\right]\)
  - Also known as decimation
  - Can be used with most operators
CNN Upsampling

- Upsampling of \((L_v, L_h) = (2,2)\), results in

\[
x(j_1, j_2) = z(\lfloor j_1/2 \rfloor, \lfloor j_2/2 \rfloor)
\]

Comments:
- Also known as pixel replication
- Output has dimensions of \((L_v N_v, L_h N_h)\)
### CNN Average Pooling

- 2D average pooling equation

\[
x(j_1, j_2) = \sum_{k_1=0}^{p-1} \sum_{k_2=0}^{p-1} \frac{1}{p^2} z(j_1 + k_1, j_2 + k_2)
\]

For \( p = 2 \), and stride of \((1,1)\), we have

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\Rightarrow
\begin{array}{cccccccc}
1 & 1 & 1 & 1/2 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1/2 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1/2 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 1/2 & 1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
2D average pooling equation

\[ x(j_1, j_2) = \sum_{k_1=0}^{p-1} \sum_{k_2=0}^{p-1} \frac{1}{p^2} z(d_1 j_1 + k_1, d_2 j_2 + k_2) \]

For \( p = 2 \) with stride of \((d_1, d_2) = (2,2)\), we have

\[ z(j_1, j_2) \]
**CNN Max Pooling**

- 2D max pooling equation

\[ x(j_1, j_2) = \max_{0 \leq k_1 < p} \max_{0 \leq k_2 < p} z(j_1 + k_1, j_2 + k_2) \]

For \( p = 2 \) and stride of \((d_1, d_2) = (1,1)\), we have

\[ z(j_1, j_2) \quad \Rightarrow \quad x(j_1, j_2) \]
2D max pooling with a stride equation

\[ x(j_1, j_2) = \max_{0 \leq k_1 < p} \max_{0 \leq k_2 < p} z(d_1 j_1 + k_1, d_2 j_2 + k_2) \]

For \( p = 2 \) and stride of \( (d_1, d_2) = (2,2) \), we have

\[ z(j_1, j_2) \]

\[ x(j_1, j_2) \]
Adjoint Gradients for CNNs

- Implementation of a CNN node
- The fast adjoint gradient
- Computing the adjoint gradient for CNN nodes
Backpropagation for CNN Nodes

- For each node you need two functions:
  - Forward propagation function:
    \[
    x \leftarrow f(z, \theta)
    \]
    You need a software implementation of the forward function.

  - Adjoint gradient function:
    \[
    [\delta, g_w, g_b] \leftarrow G(\varepsilon, z, \theta)
    \]
    You need a software implementation of this function that multiplies \( \varepsilon \) by the adjoint gradient.
**Gradient for Single Layer CNN**

- **Single layer NN:**

  \[ f_\theta(z) = \sigma(w \ast z + b) \]

  - We will need the adjoint gradient w.r.t \( \theta = (w, b) \)

  \[ \nabla_\theta f_\theta(z) = [\nabla_w f_{(w,b)}(z), \nabla_b f_{(w,b)}(z)] \]

  - And, we will also need:

  \[ \nabla_z f_\theta(z) \]
Closer Look at the Adjoint Gradient

- Each node needs a function to compute the adjoint gradient:

\[ \delta, g_w, g_b \leftarrow G(\epsilon, z, \theta) \]

\[ g = (g_w, g_b) \]

- Conceptually, this is

\[ A = \nabla_z f_\theta(z) \]
\[ B_w = \nabla_w f_\theta(z) \]
\[ B_b = \nabla_b f_\theta(z) \]
The Fast Adjoint Gradient

- Conceptually, the adjoint gradient function computes this

\[ \delta = A^t \varepsilon \]

\[ B_w = \nabla_w f_\theta(z) \]

\[ B_b = \nabla_b f_\theta(z) \]

- For input gradient, it computes

\[ A_{j,i} = \nabla_z f_\theta(z) = \frac{\partial (f_\theta(y_k))_j}{\partial z_i} \]

- You need this output.

- You’re given this input.

- But do you really need to compute this huge matrix? No!

- Big Idea: Directly compute output without computing gradient!
Adjoint Gradient of Convolution w.r.t. Input

- **Gradient of output with input:**
  \[ x_i = w_i * z_i = \sum_j w_{i-j} z_j \]
  
  So \( x = A z \) where \( A_{i,j} = w_{i-j} \)
  
  \[ \frac{\partial x_i}{\partial z_j} = A_{i,j} = w_{i-j} \]

- **Adjoint gradient:**
  \[ (A^t)_{i,j} = A_{j,i} = w_{j-i} \]
  
  \[ \delta_i = \sum_j w_{j-i} \epsilon_j = w_{-i} * \epsilon_i \]
Adjoint Gradient of Convolution w.r.t. Weights

- Gradient of output with weights:
  \[ x_i = z_i \ast w_i = \sum_{j} z_{i-j} w_j \]
  \[ x = Aw \]
  where \( \frac{\partial x_i}{\partial w_j} = A_{i,j} = z_{i-j} \)

- Adjoint gradient:
  \[ [A^t]_{i,j} = A_{j,i} = z_{j-i} \]
  \[ \delta_i = \sum_{j} z_{j-i} \varepsilon_j \]
Forward function:

\[ f(y) = \sigma(w_{(j_1,j_2),i} j_3 * z_{(j_1,j_2),i} + b j_3) \]

- The adjoint gradient \([\nabla_z f]^t \epsilon\) is given by

\[ \delta_{j_1,j_2,i} = w_{(-j_1,-j_2),i} j_3 [\nabla \sigma]^{k_1,k_2,k_3}_{(j_1,j_2),j_3} \epsilon_{k_1,k_2,k_3} \]

Fast because it never computes A!
Adjoint Gradient w.r.t. $b$

- **Forward function:**
  
  $$f(y) = \sigma \left( w_{(j_1, j_2), i} j_3 \star z_{(j_1, j_2), i} + b j_3 \right)$$

  - The adjoint gradient $[\nabla_b f]^t \epsilon$ is given by

  $$\begin{bmatrix} g_b \end{bmatrix}_{j_3} = 1_{j_1, j_2} [\nabla \sigma]_{k_1, k_2, k_3}^{j_1, j_2, j_3} \epsilon_{k_1, k_2, k_3}$$

  ![Diagram](image)

  Note: $1_{j_1, j_2} = 1$ for all $j_1$ and $j_2$
Adjoint Gradient w.r.t. \( w \)

- **Forward function:**

  \[
  f(y) = \sigma \left( w_{(j_1,j_2),i}^{j_3} \ast z_{(j_1,j_2),i}^{j_3} + b^{j_3} \right)
  \]

- The adjoint gradient \([\nabla_w f]^t \epsilon\) is given by

  \[
  [g_w]_{j_1,j_2,i,j_3} = z_{(-j_1,-j_2),i} \cdot [\nabla \sigma]^{k_1,k_2,k_3}_{(j_1,j_2),j_3} \epsilon_{k_1,k_2,k_3}
  \]