BME/ECE 695 Deep Learning
Midterm I Solution
March 11, Spring 2021

Q1.
2 Points

Rules: I understand that this is an open book exam that shall be done within the allotted time of 120 minutes. I can use my notes, and web resources. However, I will not communicate with any other person other than the official exam proctors during the exam, and I will not seek or accept help from any other persons other than the official proctors.

Upload a scan of your signature here:

Name: (4 pt) ____________________________________

Q2 Minimum of Continuous Function
(10 Points)

Let $f : [0,1]^N \Rightarrow \Re$ be a continuous function.
Does f take on a global minimum? Either prove the result or give a counter example.

Solution:

Q2

Yes, f takes on a global minimum.

Proof:
Let $\mathcal{A}$ denote the set $[0,1]^N \subset \Re^N$. Since $\mathcal{A} = [0,1]^N$ is the N-dimensional Cartesian product of the closed and bounded interval $[0,1]$, $\mathcal{A}$ is both closed and bounded. Hence, $\mathcal{A}$ is a compact set.
Given that f is a continuous function, it must attain a maximum and a minimum on a compact set. Therefore, f takes on a global minimum.

Q3 Minimum of Convex Function
(10 Points)

Let $f : \Re^N \Rightarrow \Re$ be a convex function.
Does f take on a global minimum? Either prove the result or give a counter example.

Solution:

Q3

No, f does not necessarily take on a global minimum.

Counter example:
A convex function may not achieve a minimum value over its domain.

For example,

$$f(x) = Ax + b$$

where $x \in \Re^N, A \in \Re^{1\times N}, b \in \Re$.

Q4
(10 Points)

Let $f : \Re^N \Rightarrow \Re$ be a convex function with local minimum $x^* \in \Re^N$.

Q4.1 Miminum
(5 Points)
Is x^* a global minimum? Either prove the result or give a counter example.

Q4.2
(5 Points)
Is x^* a unique local minimum? Either prove the result or give a counter example.

Solution:

Q4.1

Yes, x^* is a global minimum.

Proof:
Since $x^* \in \Re^N$ is a local minimum, then for $\alpha \in [0, 1]$ and $x \in \Re^N$ we have,

$$f(x^*) \leq f(x^* + \alpha\Delta x) = f(x^* + \alpha(x - x^*)) = f(\alpha x + (1 - \alpha)x^*)$$

By the definition of convex function,

$$f(\alpha x + (1 - \alpha)x^*) \leq \alpha f(x) + (1 - \alpha)f(x^*)$$

Hence,

$$f(x^*) \leq \alpha f(x) + (1 - \alpha)f(x^*)$$

$\Rightarrow f(x^*) \leq f(x)$, which implies that x^* is a global minimum of convex function f.

Q4.2

No, x^* is not necessarily a unique local minimum.

Counter example:
The local minimum of a convex function is not necessarily unique.

For example, considering the constant function

$$f(x) = c$$

where $x \in \Re^N$ and $c \in \Re$ is a constant, the local minimum exists but not unique because every point is a local minimum.

Q5 Gradient Descent
(35 Points)

Consider the loss function $L : \Re^2 \Rightarrow \Re$ where

$$L(x) = a(x_1 + x_2)^2 + b(x_1 - x_2)^2$$

where $a > 0$ and $b > 0$ are scalar parameters and $x = [x_1, x_2]$.

Q5.1 Miminum
(5 Points)
Prove that L convex.
Hint: You can use the "linear transform" property of convexity from the notes.

Q5.2
(5 Points)
Does $L(x)$ take on a global minimum?
Justify your answer, and if the answer is "yes", then give its value?

Q5.3
(5 Points)
Calculate the gradient $\nabla L(x)$.

Q5.4
(5 Points)
Write out the gradient descent algorithm with step size α.

Q5.5
(5 Points)
If $b = 100$ and $a = 1$, then sketch a contour plot of the function $L(x)$. Label the key dimensions and axes.

Q5.6
(5 Points)
If $b = 100$ and $a = 1$, then sketch a typical path of the gradient descent algorithm on a contour plot of the loss function.

Q5.7
(5 Points)
Explain in words why gradient descent optimization will be slow when $b >> a$.

Solution:

Q5.1

Proof:
Let $s : \Re \Rightarrow \Re$ denote the square function where $s(y) = y^2$ for $y \in \Re$, and $f : \Re^2 \Rightarrow \Re$ be

$$f(x) = as(x_1) + bs(x_2) = ax_1^2 + bx_2^2$$

Given $a > 0, b > 0$, $f(x)$ is a convex function since s is convex and f is a linear combination of the convex function.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, the linear transform of the convex function $f(x)$:

$$L(x) = f(Ax) = f(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = f(\begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}) = a(x_1 + x_2)^2 + b(x_1 - x_2)^2$$

is convex by "linear transformation" property of convexity.

Q5.2

$L(x)$ takes on a global minimum because it's continuously differentiable and convex on $\Re^2$. By solving $\Delta_x L(x) = 0$, we know the function achieves global minimum value 0 at $[x_1, x_2] = [0, 0]$.

Q5.3

The gradient is

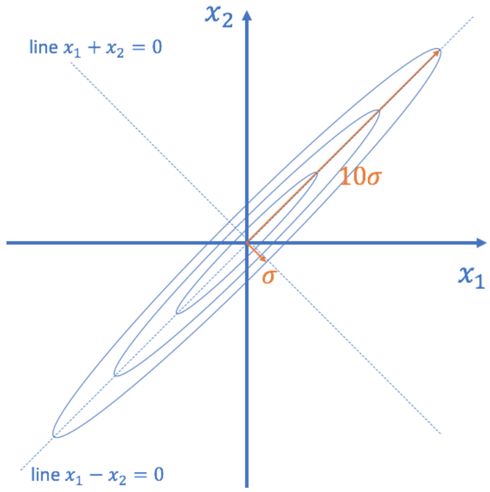
$$\begin{aligned} \Delta L(x) &= [\frac{\partial L(x)}{\partial x_1}, \frac{\partial L(x)}{\partial x_2}] \\ &= [2ax_1 + 2ax_2 + 2bx_1 - 2bx_2, 2ax_1 + 2ax_2 - 2bx_1 + 2bx_2] \\ &= [2(a+b)x_1 + 2(a-b)x_2, 2(a-b)x_1 + 2(a+b)x_2] \end{aligned} \tag{1}$$

Q5.4

Repeat until converged{
$d \leftarrow -\Delta_x L(x)$
$x \leftarrow x + \alpha d^t$
}

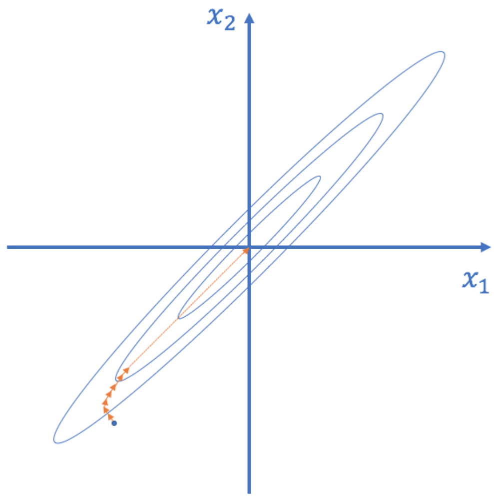
Q5.5

Given $b = 100$ and $a = 1$, the contour plot is shown below



Q5.6

An example path of gradient descent algorithm is shown below



Q5.7

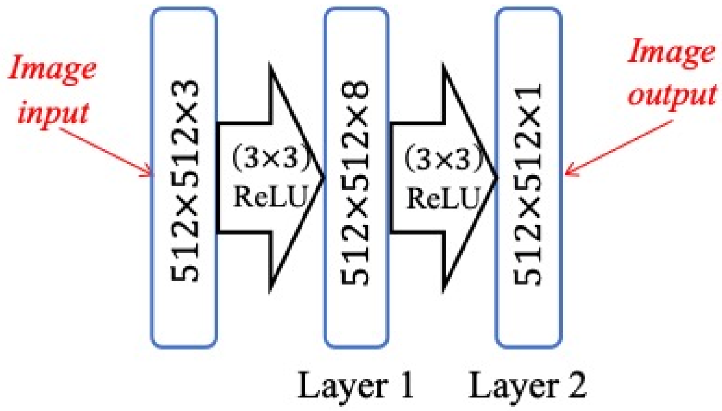
Define the following two unit vectors.

$$e_1 = \frac{[1, 1]}{\sqrt{2}}$$
$$e_2 = \frac{[1, -1]}{\sqrt{2}}$$

When $b >> a$, the gradient changes rapidly along the e_2 direction, but the gradient changes slowly along the e_1 direction. Consequently, a step size that avoids oscillations along the e_2 direction, will result in slow convergence along e_1 direction. So the convergence of the overall algorithm will be slow.

Q6 Convolutional Neural Networks
(30 Points)

Consider the following convolutional neural network pictured below with a color image as input, and a gray-scale image as output. Each layer uses a ReLu activation function, and denote the convolution kernel by w and the offsets by b.



Q6.1
(5 Points)
For layer 1, what are the shapes of the tensors w and b?

Q6.2
(5 Points)
For layer 1, what are the number of parameters?

Q6.3
(5 Points)
For layer 2, what are the shapes of the tensors w and b?

Q6.4
(5 Points)
For layer 2, what are the number of parameters?

Q6.5
(10 Points)
What are the advantages of a convolutional layer over a fully connected layer?

Solution:

Q6.1

For layer 1, the shape of tensor w is $3 \times 3 \times 3 \times 8$ and the shape of b is 8×1.

Q6.2

The total number of parameters for layer 1 is

$$3 \times 3 \times 3 \times 8 + 8 \times 1 = 224$$

Q6.3

For layer 2, the shape of tensor w is $3 \times 3 \times 8 \times 1$ and the shape of b is 1×1.

Q6.4

The total number of parameters for layer 2 is

$$3 \times 3 \times 8 \times 1 + 1 \times 1 = 73$$

Q6.5

The advantages of a convolutional layer over a fully connected layer include:

- The convolutional layer requires fewer model parameters.
- The convolutional layer requires less training data.
- The convolutional layer requires less computation and memory.
- The convolutional layer is space invariant.
- The convolutional layer typically achieves better accuracy in computer vision and image processing tasks.