Lecture 16
Group Decision Making and Aggregation of Individual Preferences

Jitesh H. Panchal

ME 597: Decision Making for Engineering Systems Design

Design Engineering Lab @ Purdue (DELP)
School of Mechanical Engineering
Purdue University, West Lafayette, IN
http://engineering.purdue.edu/delp

December 02, 2014
1. Group Decision Making
   - The Certainty Model
   - The Uncertainty Model

2. Aggregating Individuals’ Preferences
   - Arrow’s Impossibility Theorem
   - Aggregation under Certainty
   - Under Uncertainty

3. Additive Utility and Equity Considerations
Decision Maker (in this lecture): Person who wishes to incorporate the feelings, values, preferences, utilities of others into her own value assessments. The decision maker’s preferences depends on the preferences of the others.

How can a (supra) Decision Maker systematically incorporate the views of others into her own decision making framework?
Assume that the consequences $x$ of her decisions can be described in terms of attributes $X_1, X_2, \ldots, X_M$

*Overall objective*: Maximize the web-being of $N$ specified individuals

*N Lower level objectives*: Maximize the well-being of individual $i$
The Certainty Model

Decision maker’s value function: \( v \) over consequences \( x \)

Individuals’ value functions: \( v_1, v_2, \ldots, v_N \)
Find an appropriate model \( v_D \) such that

\[
v(x) = v_D[v_1(x), v_1(x), \ldots, v_N(x)]
\]

Assumptions in this form:

1. Decision maker’s preferences are entirely captured by \( v_i \)’s
2. Individual \( i \)’s preference structure is completely captured by \( v_i \)
3. Decision maker knows all the \( v_i \)’s
Decision maker’s utility function: $u$ over consequences $x$

Individuals’ utility functions: $u_1, u_2, \ldots, u_N$
Find an appropriate model such that

$$u(x) = u_D[u_1(x), u_1(x), \ldots, u_N(x)]$$

Assumptions in this form:

1. Decision maker’s preferences are entirely captured by $u_i$’s
2. Individual $i$’s preference structure is completely captured by $u_i$
3. Decision maker knows all the $u_i$’s
Example

Assume that the Decision Maker has a choice between:
- Certainty consequence $C$
- Lottery $L$ with certainty consequences $A$ and $B$

The Decision Maker ascertains that:
- For individual $i = 1$: $u_A^1 = 1$, $u_B^1 = 0$, $u_C^1 = 0.4$
- For individual $i = 2$: $u_A^2 = 0$, $u_B^2 = 1$, $u_C^2 = 0.4$

So, the Decision Maker has to choose between:
- Lottery $\langle (1, 0), (0, 1) \rangle$
- Certainty $(0.4, 0.4)$

Must consider the tradeoffs among the impacts on individuals $i = 1$ and $i = 2$. These individuals have no authority in the decision making.
Arrow’s Problem

Given the rankings of a set of alternatives by each individual in a decision making group, what should the group ranking of these alternatives be?
Arrow’s Assumptions (1)

Assumptions

A: *Complete Domain*: There are at least two individuals in the group, at least three alternatives, and a group ordering is specified for all possible individual members’ ordering.

B: *Positive Association of Social and Individual Orderings*: IF the group ordering indicates $A \succ B$, and if (1) individual’s paired comparison between alternatives other than $A$ are not changed, and (2) each individual’s paired comparison between $A$ and any other alternative either remains unchanged or is modified in $A$’s favor, THEN the group ordering must imply $A$ is still preferred to $B$.

C: *Independence of Irrelevant Alternatives*

D: *Individual’s Sovereignty*

E: *Non-dictatorship*
Assumptions

A: Complete Domain

B: Positive Association of Social and Individual Orderings

C: Independence of Irrelevant Alternatives: If an alternative is eliminated from consideration and the preference relations for the remaining alternatives remain invariant for all the group members, then the new group ordering for the remaining alternatives should be identical to the original group ordering for these same alternatives.

D: Individual's Sovereignty: For each pair of alternatives A and B, there is some set of individual orderings such that the group prefers A to B.

E: Non-dictatorship: There is no individual with the property that whenever he prefers alternative A to B, the group also prefers A to B regardless of the other individuals’ preferences.
Arrow’s Impossibility Theorem

Theorem (Arrow’s Impossibility Theorem)

Assumptions A, B, C, D, and E are inconsistent.

Interpretation: There is no procedure for combining individual rankings into a group ranking that does not explicitly address the question of interpersonal comparison of preferences!
Assume that $v_D$ exists, such that

$$v(x) = v_D[v_1(x), v_1(x), \ldots, v_N(x)]$$

Further assumptions:

1. **Preferential independence:** The attributes $\{V_i, V_j\}$ are preferentially independent of their complement $\bar{V}_{ij}$, for all $i \neq j, N \geq 3$

2. **Ordinal positive association:** Let certain alternatives $A$ and $B$ be equally preferred by the group. If $A$ is modified to $A'$ in such a manner that some individual $i$ prefers $A'$ to $A$ but all other individuals remain indifferent, then $A'$ is preferred to $B$ by the group.
Theorem

Given $N \geq 3$, Assumption 1 (preferential independence) and Assumption 2 (ordinal positive association) hold if and only if

$$v(x) = \sum_{i=1}^{N} v_i^*[v_i(x)] = \sum_{i=1}^{N} v_i^+(x)$$

where, for all $i$,

1. $v_i$ is a value function for individual $i$ scaled from 0 to 1.
2. $v_i^*$, a positive monotonic transformation of its argument $v_i$, is the Decision Maker’s value function over $V_i$, reflecting her interpersonal comparison of the individuals preferences.
3. $v_i^+$ defined as $v_i^*(v_i)$ is another value function for individual $i$ consistently scaled to reflect the Decision Maker’s interpersonal comparison of preference.
Assume the existence of $u_D$ such that

$$u(x) = u_D[u_1(x), u_2(x), \ldots, u_N(x)]$$

In addition to this, the critical condition (Harsanyi, 1955) for

$$u(x) = \sum_{i=1}^{N} \lambda_i u_i(x)$$

is:

**Assumption H**

**Assumption H:** If two alternatives, defined by probability distributions over the consequences $x$, are indifferent to each individual, then they are indifferent for the group as a whole.
Additive Independence
The set of attributes \(U_1, U_2, \ldots, U_N\) is additive independent.

Strategic Equivalence
The Decision Maker’s conditional utility function \(u_i^*\) over the attribute \(U_i\) designating individual \(i\)'s utility is strategically equivalent to individual \(i\)'s utility function \(u_i\). [Honesty Assumption]
**Theorem**

For $N \geq 2$, Assumptions 3 (additive independence) and 4 (strategic equivalence) hold if and only if

$$u(x) = \sum_{i=1}^{N} \lambda_i u_i(x)$$

where $u_i, i = 1, 2, \ldots, N,$ is a utility function for individual $i$ scaled from 0 to 1, the $\lambda_i$'s are positive scaling constants, and $x$ is a consequence.

Assumption H and the pair of assumptions (3 and 4) are equivalent. The Decision Maker’s interpersonal comparison of the individual’s preferences is required to assess the $\lambda_i$ scaling factors.
Assumption 5: Utility Independence

Attribute $U_i$, $i = 1, 2, \ldots, N$, is utility independent of the other attributes $\bar{U}_i$. This implies

$$u_D(u_1, \ldots, u_i, \ldots, u_N) = g_i(\bar{u}_i) + f_i(\bar{u}_i)u_i^*(u_i) \quad \forall i$$

Theorem

For $N \geq 2$, Assumptions 4 (strategic equivalence) and 5 (utility independence) imply

$$u(x) = u_D(u_1, u_2, \ldots, u_N)$$

$$u(x) = \sum_{i=1}^{N} \lambda_i u_i(x) + \sum_{i=1, j>i}^{N} \lambda_{ij} u_i(x)u_j(x) + \cdots + \lambda_{12\ldots N} u_1(x)u_2(x)\ldots u_N(x)$$
Assumption 1A: Preferential Independence

The attributes \( \{U_i, U_j\} \) are preferentially independent of their complement \( \bar{U}_{ij} \) for all \( i \neq j, \ N \geq 3 \)

Theorem

For \( N \geq 3 \), Assumptions 1 or 1A (preferential independence), 4 (strategic equivalence) and 5 (utility independence) imply

\[
\begin{align*}
  u(x) &= u_D(u_1, u_2, \ldots, u_N) \\
  u(x) &= \sum_{i=1}^{N} \lambda_i u_i(x) + \lambda \sum_{i=1, j>i}^{N} \lambda_i \lambda_j u_i(x) u_j(x) + \ldots \\
  &\quad + \lambda^{N-1} \lambda_1 \lambda_2 \ldots \lambda_N u_1(x) u_2(x) \ldots u_N(x)
\end{align*}
\]

where \( u \) and the \( u_i \)'s are scaled from 0 to 1, the \( \lambda \)'s are scaling constants, \( 0 < \lambda_i < 1 \) for all \( i \), and \( \lambda > -1 \).
Consider an example: A Decision Maker is interested in the preferences of two individuals.

\[ u(x) = u_D(u_1(x), u_2(x)) = 0.5u_1(x) + 0.5u_2(x) \]

Let (0.4, 0.6) designate the alternative where \( u_1 = 0.4 \) and \( u_2 = 0.6 \)

Consider the following alternatives:

- Alternative A: (1, 0)
- Alternative B: ⟨(1, 0), (0, 1)⟩
- Alternative C: ⟨(1, 1), (0, 0)⟩

Alternatives B and C are \textit{a priori} equitable, whereas alternative A is not. Alternative C is more equitable in terms of \textit{posterior} equity.
Another utility function:

\[ u(x) = 0.4u_1(x) + 0.4u_2(x) + 0.2u_1(x)u_2(x) \]

Expected utilities for alternatives A, B, and C are 0.4, 0.4, and 0.5 respectively.

Accounts for posterior equity.
Summary

1. Group Decision Making
   - The Certainty Model
   - The Uncertainty Model

2. Aggregating Individuals’ Preferences
   - Arrow’s Impossibility Theorem
   - Aggregation under Certainty
   - Under Uncertainty

3. Additive Utility and Equity Considerations