Lecture 06
Single Attribute Utility Theory

Jitesh H. Panchal

ME 597: Decision Making for Engineering Systems Design

*Design Engineering Lab @ Purdue (DELP)*
School of Mechanical Engineering
Purdue University, West Lafayette, IN
http://engineering.purdue.edu/delp

September 17, 2014
Lecture Outline

1. Decision Making under Uncertainty
   - Alternate Approaches to Risky Choice Problem
   - Motivation for Using Utility Theory

2. Uni-dimensional Utility Theory
   - Fundamentals of Utility Theory
   - Qualitative Characteristics of Utility

3. Attitudes to Risk
   - Risk Averse and Risk Prone
   - Measuring Risk Aversion

4. A Procedure for Assessing Utility Functions

The Structure of a Design Decision

Decision \( A_1 \)

\( A_1 \)\( \rightarrow \) Outcomes

\( p_{11} \)\( \rightarrow \) \( O_{11} \) \( \rightarrow \) \( U(O_{11}) \)

\( p_{1k} \)\( \rightarrow \) \( O_{1k} \) \( \rightarrow \) \( U(O_{1k}) \)

Select \( A_i \)

Decision \( A_2 \)

\( A_2 \)\( \rightarrow \) Outcomes

\( p_{21} \)\( \rightarrow \) \( O_{21} \) \( \rightarrow \) \( U(O_{21}) \)

\( p_{1k} \)\( \rightarrow \) \( O_{2k} \) \( \rightarrow \) \( U(O_{2k}) \)

Decision \( A_n \)

\( A_n \)\( \rightarrow \) Outcomes

\( p_{n1} \)\( \rightarrow \) \( O_{n1} \) \( \rightarrow \) \( U(O_{n1}) \)

\( p_{nk} \)\( \rightarrow \) \( O_{nk} \) \( \rightarrow \) \( U(O_{nk}) \)

Alternatives \( \rightarrow \) Outcomes \( \rightarrow \) Preferences \( \rightarrow \) Choice

Slide courtesy: Chris Paredis
Focus of this Lecture

Problem Statement

Choose among alternatives $A_1, A_2, \ldots, A_m$, each of which will eventually result in a consequence described by one attribute $X$.

Decision maker *does not* know exactly what consequence will result from each alternative.

But he/she *can* assign probabilities to the various consequences that might result from any alternative.
Alternate Approaches to Risky Choice Problem: a) Probabilistic dominance

Figure: 4.2 on page 135 (Keeney and Raiffa)
Consider the following acts

- Act A: Earn $100,000 for sure
- Act B: Earn $200,000 or $0, each with probability 0.5
- Act C: Earn $1,000,000 with probability 0.1 or $0 with probability 0.9
- Act D: Earn $200,000 with probability 0.9 or lose $800,000 with probability 0.1

The expected amount earned is exactly $100,000. But not all acts are equally desirable.
Alternate Approaches to Risky Choice Problem:
c) Consideration of mean and variance

One possibility is to consider variance, in addition to the expected value of the outcome.

But, Acts C and D have the same mean and variance:

- Act C: Earn $1,000,000 with probability 0.1 or $0 with probability 0.9
- Act D: Earn $200,000 with probability 0.9 or lose $800,000 with probability 0.1

Are these acts equally preferred?

Therefore, any measure that considers mean and variance only cannot distinguish between these two acts.

Considering mean and variance imposes additional problem of finding relative preference between them.
Primary Motivation for using Utility Theory

IF an appropriate utility is assigned to each possible consequence, AND the expected utility of each alternative is calculated,

THEN the best course of action is the alternative with the highest expected utility.
Assume \( n \) consequences labeled \( x_1, x_2, \ldots, x_n \) such that \( x_i \) is less preferred than \( x_{i+1} \)

\[
x_1 \prec x_2 \prec x_3 \prec \cdots \prec x_n
\]

The decision maker is asked to state preferences about two acts \( a' \) and \( a'' \), where

1. Act \( a' \) will result in consequence \( x_i \) with probability \( p'_i \) for \( i = 1..n \)
2. Act \( a'' \) will result in consequence \( x_i \) with probability \( p''_i \) for \( i = 1..n \)
Assume that for each $i$, the decision maker is indifferent between the following options:

**Certainty Option:** Receive $x_i$

**Risky Option:** Receive $x_n$ with probability $\pi_i$ and $x_1$ with probability $(1 - \pi_i)$. This option is denoted as $\langle x_n, \pi_i, x_1 \rangle$

Clearly,

\[
\begin{align*}
\pi_1 &= 0 \\
\pi_n &= 1 \\
\pi_1 &< \pi_2 < \pi_3 < \cdots < \pi_n
\end{align*}
\]
π_i’s can be thought of as numerical scaling of x’s.

\[ \pi_1 < \pi_2 < \pi_3 < \cdots < \pi_n \]

and

\[ x_1 < x_2 < x_3 < \cdots < x_n \]

**Fundamental Result of Utility Theory**

The expected value of the π’s can be used to numerically scale probability distributions over the x’s.

The expected π scores for acts \( a' \) and \( a'' \) are as follows:

\[ \bar{\pi}' = \sum_i p'_i \pi_i \quad \text{and} \quad \bar{\pi}'' = \sum_i p''_i \pi_i \]

Act \( a' \) \equiv giving the decision maker a \( \bar{\pi}' \) chance at \( x_n \) and \( 1 - \bar{\pi}' \) chance at \( x_1 \)

Act \( a'' \) \equiv giving the decision maker a \( \bar{\pi}'' \) chance at \( x_n \) and \( 1 - \bar{\pi}'' \) chance at \( x_1 \)

Now, we can rank order acts \( a', a'' \) in terms of \( \bar{\pi}', \bar{\pi}'' \)
Transforming $\pi$’s into $u$’s using a positive linear transformation

$$u_i = a + b\pi_i, \quad b > 0, \quad i = 1, \ldots, n$$

Then,

$$u_1 < u_2 < \cdots < u_n$$

The expected $u$ values rank order $a'$ and $a''$ in the same way as the expected $\pi$ values

$$\bar{u}' = \sum_i p'_i u_i = \sum_i p'_i (a + b\pi_i) = a + b\bar{\pi}'$$

**Essence of the problem**

How can appropriate $\pi$ values be assessed in a responsible manner?
Define:
\( x^o \) as a least preferred consequence, and 
\( x^* \) as a most preferred consequence. Assign

\[
u(x^*) = 1 \quad \text{and} \quad u(x^o) = 0
\]

For each other consequence \( x \), assign a probability \( \pi \) such that \( x \) is indifferent to the lottery \( \langle x^*, \pi, x^o \rangle \). Note that the expected utility of the lottery is:

\[
u(x) = \pi u(x^*) + (1 - \pi) u(x^o) = \pi
\]

Continue for all \( x \)'s (or fit a curve).
Qualitative Characteristics of Utility

1. Monotonicity
2. Certainty equivalence
3. Strategic equivalence
Qualitative Characteristics of Utility: Monotonicity

**Definition (Monotonicity)**

For a monotonically increasing utility function

\[ x_1 > x_2 \iff u(x_1) > u(x_2) \]

For a monotonically decreasing utility function

\[ x_1 > x_2 \iff u(x_1) < u(x_2) \]
Qualitative Characteristics of Utility: Certainty Equivalence

Assume lottery \( L \) yields consequences \( x_1, x_2, \ldots, x_n \) with probabilities \( p_1, p_2, \ldots, p_n \).

Define:
\( \tilde{x} \): Uncertain consequence of lottery (i.e., random variable)
\( \bar{x} \): Expected consequence

\[
\bar{x} \equiv E(\tilde{x}) = \sum_{i=1}^{n} p_i x_i
\]

Definition (Certainty equivalence)
A certainty equivalent of lottery \( L \) is the amount \( \hat{x} \) such that the decision maker is indifferent between \( L \) and the amount \( \hat{x} \) for certain.

\[
u(\hat{x}) = E[u(\tilde{x})], \quad \text{or} \quad \hat{x} = u^{-1} E u(\tilde{x})
\]
If $x$ is a continuous variable, the associated uncertainty is described using a probability density function, $f(x)$. Then,

$$\bar{x} \equiv E(\tilde{x}) = \int xf(x)dx$$

The certainty equivalent $\hat{x}$ is a solution to

$$u(\hat{x}) = E[u(\tilde{x})] = \int u(x)f(x)dx$$
Definition (Strategic equivalence)

Two utility functions, $u_1$ and $u_2$, are strategically equivalent ($u_1 \sim u_2$) if and only if they imply the same preference ranking for any two lotteries.

If two utility functions are strategically equivalent, the certainty equivalents of two lotteries must be the same. Therefore,

$$u_1 \sim u_2 \Rightarrow u_1^{-1} Eu_1(\tilde{x}) = u_2^{-1} Eu_2(\tilde{x}), \quad \forall \tilde{x}$$
For some constants \( h \) and \( k > 0 \), if

\[
    u_1(x) = h + ku_2(x), \quad \forall x
\]

then \( u_1 \sim u_2 \)

**Theorem**

*If \( u_1 \sim u_2 \), there exists two constants \( h \) and \( k > 0 \) such that*

\[
    u_1(x) = h + ku_2(x), \quad \forall x
\]

**Example:** \( u(x) = a + bx \sim x, b > 0 \)

We can show that if the utility function is linear, the certainty equivalent for any lottery is equal to the expected consequence of that lottery.
Consider a lottery \( \langle x', 0.5, x'' \rangle \) whose expected consequence is \( \bar{x} = \frac{x' + x''}{2} \).
Assume that the decision maker is asked to choose between \( \bar{x} \) for certain and the lottery \( \langle x', 0.5, x'' \rangle \).
Note: Both options have the same expected consequence.

If the decision maker prefers the certain outcome \( \bar{x} \), then the decision maker prefers to avoid risks \( \Rightarrow \) Risk Averse.

**Definition (Risk Aversion)**

A decision maker is risk averse if he prefers the expected consequence of any non-degenerate lottery to that lottery.
Risk Aversion - An Illustration

Let the possible consequences of any lottery are represented by $\tilde{x}$, a decision maker is risk averse if, for all nondegenerate lotteries, utility of expected consequence is greater than expected utility of lottery, i.e.,

$$u[E(\tilde{x})] > E[u(\tilde{x})]$$

**Theorem**

A decision maker is risk averse if and only if his utility function is concave.

**Corollary**

A decision maker who prefers the expected consequence of any 50-50 lottery $\langle x', 0.5, x'' \rangle$ to the lottery itself is risk averse.
Risk Prone

Definition (Risk Prone)

A decision maker is risk prone if he prefers any non-degenerate lottery to the expected consequence of that lottery.

\[ u[E(\tilde{x})] < E[u(\tilde{x})] \]
**Definition (Risk Premium of a lottery)**

The risk premium \( RP(\tilde{x}) \) of a lottery \( \tilde{x} \) is its expected value \( (\bar{x}) \) minus its certainty equivalent \( (\hat{x}) \).

\[
RP(\tilde{x}) = \bar{x} - \hat{x} = E(\tilde{x}) - u^{-1} Eu(\tilde{x})
\]

**Figure:** 4.5 on page 152 (Keeney and Raiffa)
Theorem

For increasing utility functions, a decision maker is risk averse if and only if his risk premium is positive for all nondegenerate lotteries.

The risk premium is the amount of the attribute that a (risk averse) decision maker is willing to “give up” from the average to avoid the risks associated with the particular lottery.
Measuring Risk Aversion

Risk Averse and Risk Prone

A Procedure for Assessing Utility Functions

Figure: 4.9 on page 159 (Keeney and Raiffa)
A Measure of Risk Aversion

Definition (Risk aversion)

The local risk aversion at $x$, written $r(x)$, is defined by

$$r(x) = -\frac{u''(x)}{u'(x)}$$

- $r(x) > 0 \Rightarrow$ Risk Averse
- $r(x) < 0 \Rightarrow$ Risk Prone

Characteristics of this measure:

1. It indicates whether the utility function is risk averse or risk prone
2. Shows equivalence between two strategically equivalent utility functions
Local Risk Aversion - Some Results

Theorem

Two utility functions are strategically equivalent if and only if they have the same risk-aversion function.

Theorem

If \( r \) is positive for all \( x \), then \( u \) is concave and the decision maker is risk-averse.

Theorem

If \( r_1(x) > r_2(x) \) for all \( x \), then \( \pi_1(x, \tilde{x}) > \pi_2(x, \tilde{x}) \) for all \( x \) and \( \tilde{x} \).
Let $x$ denote a decision maker’s endowment of a given attribute $X$. Now, add to $x$ a lottery $\tilde{x}$ involving only a small range of $X$ with an expected value of zero. Let the risk premium of this lottery be $\pi(x, \tilde{x})$.

What happens to $\pi(x, \tilde{x})$ as $x$ increases?

*Example of decreasingly risk averse:* As a person’s assets increase, they are only willing to pay a smaller risk premium for a given task (as people become richer, they can better afford to take a specific risk).
Constant, Decreasing and Increasing Risk Aversion

Implication

Many of the traditional candidates for a utility function (e.g., exponential and quadratic) are not appropriate for a decreasingly risk averse decision maker.

Theorem

The risk aversion $r$ is constant if and only if $\pi(x, \tilde{x})$ is a constant function of $x$ for all $\tilde{x}$.

Theorem

$$u(x) \sim -e^{-cx} \iff r(x) \equiv c > 0 \quad (constant \ risk \ aversion)$$
$$u(x) \sim x \iff r(x) \equiv 0 \quad (risk \ neutrality)$$
$$u(x) \sim e^{-cx} \iff r(x) \equiv c < 0 \quad (constant \ risk \ proneness)$$
Theorem

For non monotonic preferences, a decision maker is risk averse [risk prone] if and only if his utility function is concave [convex].

![Utility Functions Diagram](https://via.placeholder.com/150)

**Figure:** 4.18 on page 188 (Keeney and Raiffa)

For non-monotonic utility functions, the certainty equivalent is not necessarily unique. The risk premium and measure of risk aversion cannot be usefully defined.
A Procedure for Assessing Utility Functions

1. Preparing for assessment.
2. Identifying the relevant quality characteristics.
3. Specifying quantitative restrictions.
4. Choosing a utility function.
5. Checking for consistency.
1. Preparing for Assessment

It is important to acquaint the decision maker with the framework that we use in assessing the utility function.

Educate the decision maker (not bias him/her) and hopefully force him/her to think about his/her preferences.
2. Identifying the Relevant Quality Characteristics

1. Determine monotonicity
2. Determine whether the decision maker is risk averse, risk neutral, or risk prone
3. Determine whether the decision maker is increasingly, decreasingly, or constantly risk averse.
3. Specifying Quantitative Restrictions

Fixing utilities for a few points on the utility function.
Involves determining the certainty equivalents of a few 50-50 lotteries.
A five point assessment procedure for utility functions.

\[
\begin{align*}
    u(x_{0.5}) &= \frac{1}{2}u(x_1) + \frac{1}{2}u(x_0) \\
    u(x_{0.75}) &= \frac{1}{2}u(x_1) + \frac{1}{2}u(x_{0.5}) \\
    u(x_{0.25}) &= \frac{1}{2}u(x_0) + \frac{1}{2}u(x_{0.5})
\end{align*}
\]

Figure: 4.22 on page 195 (Keeney and Raiffa)
4. Choosing a Utility Function

Find a parametric family of utility functions that possesses the relevant characteristics (such as risk aversion) previously specified for the decision maker.

Using the quantitative assessments, try to find a specific member of that family that is appropriate for the decision maker.

An example of monotonically increasing, decreasingly risk averse utility function:

\[
u(x) = h + k(-e^{-ax} - be^{-cx}), \quad a, b, c, \text{ and } k \geq 0\]
5. Checking for Consistency

Goal: to uncover discrepancy in a utility function.
Other Considerations for Using the Utility Function

Simplifying the expected utility calculations
Parametric/sensitivity analysis
References


THANK YOU!