Module 12
Cumulative Prospect Theory

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Decision Making in Engineering Design

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Module Outline

1. Ingredients
   - 1. Probability Weighting
   - 2. Rank Dependent Utility
   - 3. Reference Dependence
2. Cumulative Prospect Theory

Tversky, A., Kahneman, D., 1992, "Advances in Prospect Theory: Cumulative Representation of Uncertainty", *Journal of Risk and Uncertainty*, Vol. 5, pp. 297-323.

Wakker, P. P., 2010, *Prospect Theory: For Risk and Ambiguity*, Cambridge University Press.

Motivation for Prospect Theory

Phenomena that violate the normative theory of expected utility maximization:

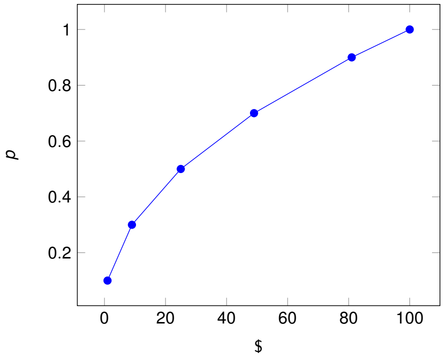
- Framing effects
- Nonlinearity of preferences in terms of probabilities (Allias' paradox)
- Source dependence (Ellsberg paradox: known vs. unknown probabilities)
- Risk seeking (e.g., People prefer small probability of winning a large prize)
- Loss aversion (asymmetry between losses and gains)

Ingredients

Assessing Utility Functions

Questions to assess a utility function:

- $\langle \$100, 0.1, \$0 \rangle \sim \boxed{???}$
- $\langle \$100, 0.3, \$0 \rangle \sim \boxed{???}$
- $\langle \$100, 0.5, \$0 \rangle \sim \boxed{???}$
- $\langle \$100, 0.7, \$0 \rangle \sim \boxed{???}$
- $\langle \$100, 0.9, \$0 \rangle \sim \boxed{???}$



Utility describes both sensitivity to money, and attitude towards risk!

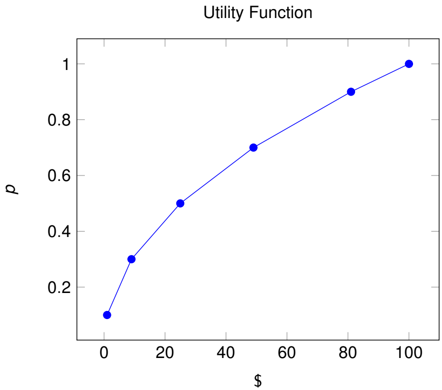
Psychologists' view

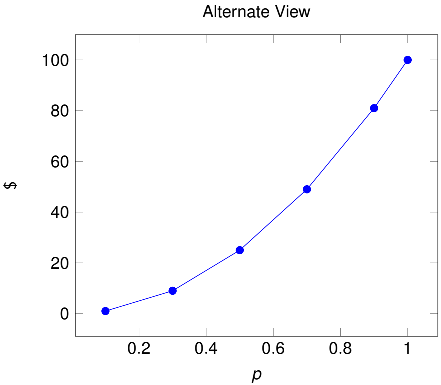
The perception of money will be approximately linear for moderate amounts of money.

Psychologists have suggested that risk attitude will have more to do with **how people feel about probability** than how they feel about money.

Instead of nonlinearly weighing outcomes $E(U) = \sum_i p_i U(x_i)$, what if we weigh probabilities $\sum_i w(p_i) x_i$.

Alternate View Explaining the Same Data Set





Lottery: $\langle \$100, 0.1, \$0 \rangle \sim \$1$

$\implies 0.1U(\$100){+}0.9U(\$0) = U(\$1)$

Lottery: $\langle \$100, 0.1, \$0 \rangle \sim \$1$

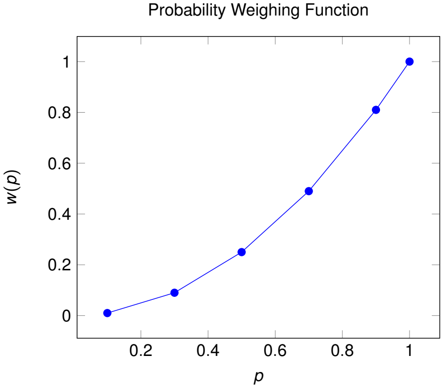
$\implies w(0.1)\$100 + w(0.9)\$0 = \$1$

1. Probability Weighing Function

- $w(0.1) = 1/100 = 0.01$
- $w(0.3) = 9/100 = 0.09$
- $w(0.5) = 25/100 = 0.25$
- $w(0.7) = 49/100 = 0.49$
- $w(0.9) = 81/100 = 0.81$

These weights can be interpreted as “perceptions of probability”. The perception of $p = 0.7$ is less than 0.7.

This is one way of looking at risk aversion.



A plausible descriptive model: Decisions are made by maximizing $\sum_i w(p_i)x_i$

Expected Value in Multi-outcome Prospects

Expected value = $p_1 x_1 + p_2 x_2 + \ldots + p_n x_n$
Assume $x_1 > x_2 > \ldots > x_n$

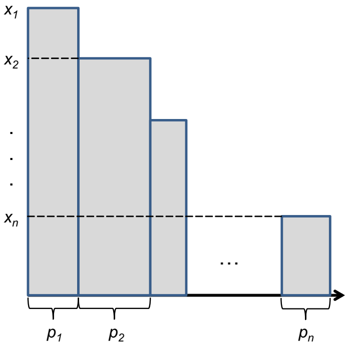


Figure: 5.2.1 on Page 150 (Wakker)

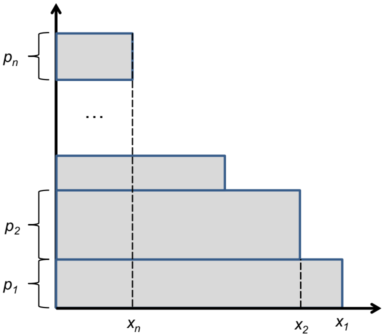


Figure: 5.2.2b on Page 150 (Wakker)

Multi-outcome Prospects

Expected Utility as a Non-linear Transformation of Value

Expected value =

$$p_1 x_1 + p_2 x_2 + \ldots + p_n x_n$$

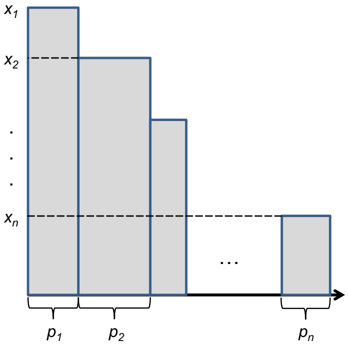


Figure: 5.2.1 on Page 150 (Wakker)

Expected utility =

$$p_1 U(x_1) + p_2 U(x_2) + \ldots + p_n U(x_n)$$

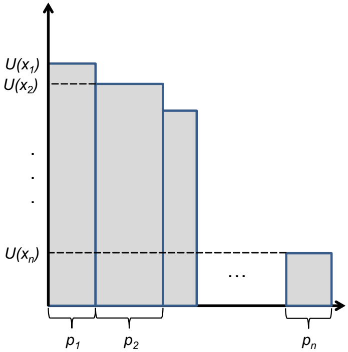


Figure: 5.2.3 on Page 151 (Wakker)

Multi-outcome Prospects

Expected Utility as a Non-linear Transformation of Probabilities

Expected value =
$p_1x_1 + p_2x_2 + \ldots + p_nx_n$

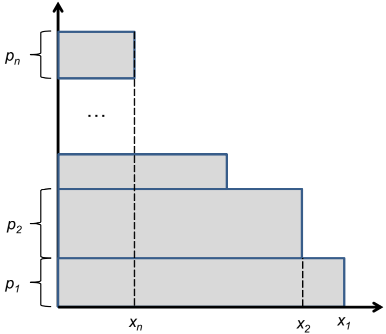


Figure: 5.2.2b on Page 150 (Wakker)

Value of the prospect:
$w(p_1)x_1 + w(p_2)x_2 + \ldots + w(p_n)x_n$

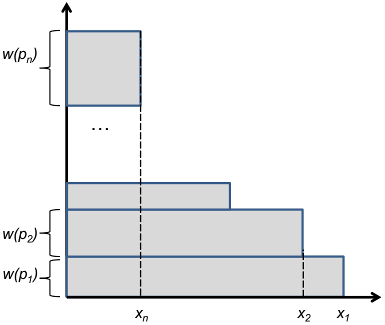


Figure: 5.2.5 on Page 152 (Wakker)

Challenge: Violation of Stochastic Dominance

Theoretical Anomaly

$w(p_1)x_1 + w(p_2)x_2 + \ldots + w(p_n)x_n$ violates stochastic dominance for non-linear $w()$ because

$$w(p_1 + p_2) \neq w(p_1) + w(p_2)$$

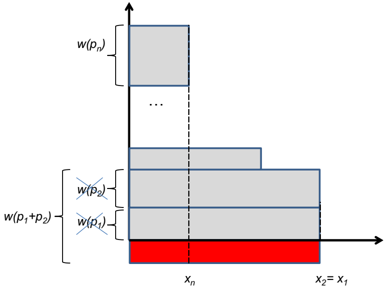


Figure: 5.3.1c on Page 154 (Wakker)

2. Rank Dependent Utility

General Idea of Rank Dependent Utility

Good-news probabilities (ranks) are sometimes more relevant to decisions than the probabilities of fixed outcomes.

Example:

# card	1-20	21-40	41-60	61-80	81-100
$-gain	$80	$60	$40	$20	$0
Probability	0.2	0.2	0.2	0.2	0.2

Good news probability (rank) of getting more than $20 = 0.6

Relevance of Good News Probabilities to Decisions

Original Lottery

# card	1-20	21-40	41-60	61-80	81-100
$-gain	$80	$60	$40	$20	$0
Probability	0.2	0.2	0.2	0.2	0.2
Good news prob. (Rank)	0	0.2	0.4	0.6	0.8

Proposed Change 1

The **separate-outcome probability** of receiving $40 is increased by changing the outcome on one card. Would you like this change?

Relevance of Good News Probabilities to Decisions

Original Lottery:

# card	1-20	21-40	41-60	61-80	81-100
$-gain	$80	$60	$40	$20	$0
Probability	0.2	0.2	0.2	0.2	0.2
Good news prob. (Rank)	0	0.2	0.4	0.6	0.8

Proposed Change 2

The **good-news probability** of receiving more than $20 is increased by changing the outcome on one card. Would you like this change?

Many studies have shown that for the happiness derived from income, the rank in society, i.e., the percentage of people who are richer, is important.

Probabilistic dominance

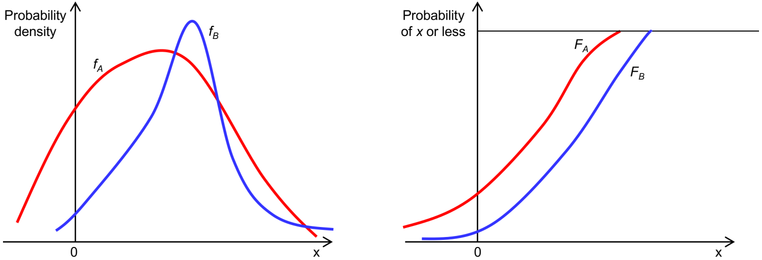


Figure: 4.2 on page 135 (Keeney and Raiffa)

Expressing Expected Utility in Terms of Ranks

\$-gain	\$80	\$60	\$40	\$20	\$0
Probability	0.2	0.2	0.2	0.2	0.2
Rank	0	0.2	0.4	0.6	0.8

$$
\begin{aligned}
EU &= 0.2U(\$80) + 0.2U(\$60) + 0.2U(\$40) + 0.2U(\$20) + 0.2U(\$0) \\
&= [0.2 - 0]U(\$80) + [0.4 - 0.2]U(\$60) + [0.6 - 0.4]U(\$40) \\
&\quad + [0.8 - 0.6]U(\$20) + [1.0 - 0.8]U(\$0) \\
&= \sum_i \underbrace{[R(x_{i+1}) - R(x_i)]}_{\text{difference in ranks}} U(x_i)
\end{aligned}
$$

where rank $R(x_i) = p_1 + p_2 + \ldots + p_{i-1}$
and $x_1 > x_2 > \ldots > x_n$

Rank-dependent Utility (RDU)

Rank Dependent Utility (RDU)

Under RDU, the weight of a utility is the difference between two **transformed** ranks.

$$RDU = \sum_i \underbrace{[w(R(x_{i+1})) - w(R(x_i))]}_{\text{difference in transformed ranks}} U(x_i)$$

Note: The weight of an outcome depends not only on its probability of occurring but also its rank.

Rank Dependent Utility

Example

Assume that $w(R) = R^2$ and $U(x) = x$

$-gain	$80	$50	$10
Probability	0.2	0.3	0.5
Rank (R)	0	0.2	0.5
w(R)	0	0.04	0.25

Decision weights:

- Decision weight of 80 = $w(0.2) - w(0) = 0.04 - 0 = 0.04$
- Decision weight of 50 = $w(0.5) - w(0.2) = 0.25 - 0.04 = 0.21$
- Decision weight of 10 = $w(1) - w(0.5) = 1 - 0.25 = 0.75$

$$\begin{aligned} RDU &= [w(0.2) - w(0)]U(80) + [w(0.5) - w(0.2)]U(50) + [w(1) - w(0.5)]U(10) \\ &= 0.04 \times 80 + 0.21 \times 50 + 0.75 \times 10 \\ &= 21.2 \end{aligned}$$

3. Reference Dependence

Differences in Attitudes towards Gains and Losses

Decision 1

Option A: Get \$50 for sure
Option B: Lottery $\langle 100, 0.5, 0 \rangle$

Are you risk averse or risk prone?

Decision 2

Option C: Lose \$50 for sure
Option D: $\langle -100, 0.5, 0 \rangle$

Are you risk averse or risk prone?

Reference Dependence

You just found \$100 lying on the ground.

Decision 2 revisited

Option C: Lose \$50 for sure
Option D: $\langle -100, 0.5, 0 \rangle$

People are more sensitive to losses than to gains.

Mental accounting [Thaler]

Loss Aversion

Utility functions considering loss aversion:

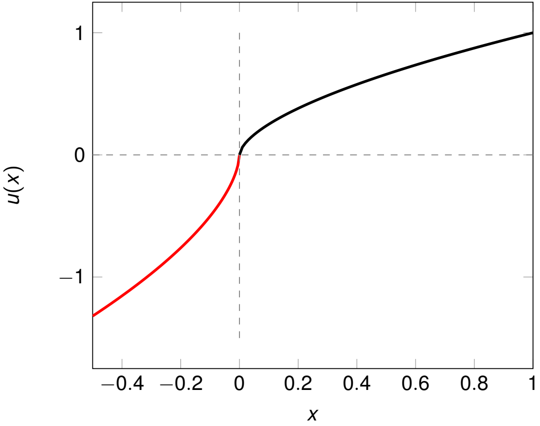
$$U(x) = \begin{cases} u(x) & \text{for } x \geq 0 \\ \lambda u(x) & \text{for } x < 0 \end{cases}$$

For loss aversion, $\lambda > 1$.
$\lambda = 2$ means that the pain for loss is twice the happiness for gain.

Example of utility function of this form

$$U(x) = \begin{cases} x^{\alpha} & \text{for } x \geq 0 \\ -\lambda(-x)^{\beta} & \text{for } x < 0 \end{cases}$$

Loss Aversion



Cumulative Prospect Theory

Cumulative Prospect Theory

Assume one fixed reference outcome \$0 (status quo). Positive outcomes are *gains*, negative outcomes are *losses*. Consider outcomes x_i with probabilities p_i, ordered as follows:

$$\underbrace{x_1 \geq \ldots \geq x_k}_{\text{Gains}} \geq 0 \geq \underbrace{x_{k+1} \geq \ldots \geq x_n}_{\text{Losses}}$$

According to prospect theory, individuals maximize

$$PT(x) = \sum_{i=1}^{n} \pi_i U(x_i)$$

where π_i are the non-negative “decision weights”.

Cumulative Prospect Theory

The decision weights are defined as follows:

- For Gains: $\pi_i = w^+(p_i + g) - w^+(g)$
- For losses: $\pi_i = w^-(p_i + l) - w^-(l)$

where g is the gain rank, and l is the loss rank.

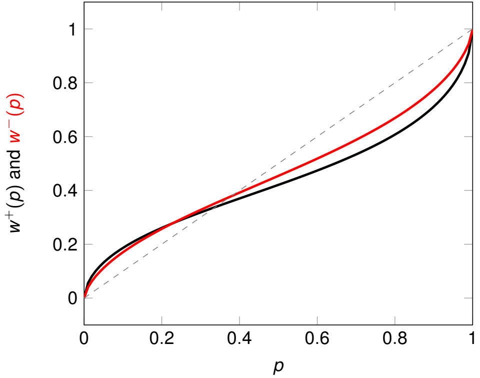
Note: two different weighting functions are used: w^+ for gains and w^- for losses.

In prospect theory, the probability weighting functions are subjective parameters that characterize the decision maker.

Weighing Functions based on Behavioral Data

For gains: $w^{+}(p) = \dfrac{p^{\gamma}}{(p^{\gamma} + (1-p)^{\gamma})^{1/\gamma}}$ and $\gamma = 0.61$

For losses: $w^{-}(p) = \dfrac{p^{\delta}}{(p^{\delta} + (1-p)^{\delta})^{1/\delta}}$ and $\delta = 0.69$



Cumulative Prospect Theory

Example: Game of Chance

Game of Chance:

Die roll	1	2	3	4	5	6
$-gain	-$1	$2	-$3	$4	-$5	$6
Probability	1/6	1/6	1/6	1/6	1/6	1/6

Ordered Consequences:

$-gain	$6	$4	$2	-$1	-$3	-$5
Probability	1/6	1/6	1/6	1/6	1/6	1/6

Gains:

$-gain	$6	$4	$2	$0
Prob.	1/6	1/6	1/6	1/2
Rank	0	1/6	1/3	1/2
π_i	$w^+(\frac{1}{6}) - w^+(0)$	$w^+(\frac{1}{3}) - w^+(\frac{1}{6})$	$w^+(\frac{1}{2}) - w^+(\frac{1}{3})$	

Cumulative Prospect Theory

Example: Game of Chance

Losses:

\$-Gain	–\$5	–\$3	–\$1	\$0
Prob.	1/6	1/6	1/6	1/2
Rank	0	1/6	1/3	1/2
π_i	$w^-(\frac{1}{6}) - w^-(0)$	$w^-(\frac{1}{3}) - w^-(\frac{1}{6})$	$w^-(\frac{1}{2}) - w^-(\frac{1}{3})$	

$$
\begin{aligned}
PT(x) = &U(6)[w^+(1/6) - w^+(0)] \\
&+ U(4)[w^+(1/3) - w^+(1/6)] \\
&+ U(2)[w^+(1/2) - w^+(1/3)] \\
&+ U(-5)[w^-(1/6) - w^-(0)] \\
&+ U(-3)[w^-(1/3) - w^-(1/6)] \\
&+ U(-1)[w^-(1/2) - w^-(1/3)]
\end{aligned}
$$

Combination of Probability Weighting and Utility Curvature

- Risk aversion for moderate- and high-probability gains
- Risk seeking for moderate and high probability losses
- Risk seeking for small-probability gains
- Risk aversion for small-probability losses

Summary

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   - 1. Probability Weighting
   - 2. Rank Dependent Utility
   - 3. Reference Dependence

2. Cumulative Prospect Theory

References

1. Tversky, A., Kahneman, D., 1992, “Advances in Prospect Theory: Cumulative Representation of Uncertainty”, *Journal of Risk and Uncertainty*, Vol. 5, pp. 297-323
2. Wakker, P. P., 2010, *Prospect Theory: For Risk and Ambiguity*, Cambridge University Press.
3. `http://psych.fullerton.edu/mbirnbaum/calculators/cpt_calculator.htm`