Module 07
Multi-attribute Utility Theory

Dr. Jitesh H. Panchal

Decision Making in Engineering Design

PURDUE MECHANICAL ENGINEERING

School of Mechanical Engineering
Purdue University, West Lafayette, IN

Learning Objectives

Learning Objectives for Module 07:

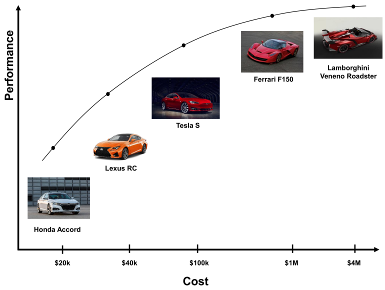
- Extensions of Utility theory to more than two attributes.
- Approaches to Multiattribute Utility assessment without specifying the functional form of Utility functions.
- Conditional, Mutual, and Additive Utility independence conditions, and their implications on the functional form of Utility functions.
- Steps in the multiattribute assessment procedure.

	Single Attribute	Multiple Attributes
Certainty	I	II
Uncertainty	III	IV

Keeney, R. L. and H. Raiffa (1993). *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. Cambridge, UK, Cambridge University Press. Chapters 5 and 6.

What have we covered so far?

(a) Deterministic scenario



Tradeoff problem (Module 05)

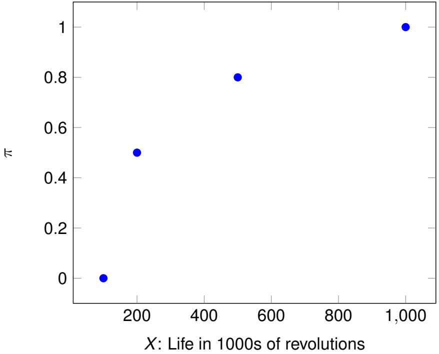
How much achievement on Objective 1 is the decision maker willing to give up in order to improve achievement on Objective 2 by some fixed amount?

What have we covered so far?

(b) Probabilistic scenario

Module 06

How much of an attribute is a (risk averse) decision maker willing to "give up" from the average to avoid the risks associated with a lottery?



Lottery questions such as

$$\langle x_n, \boxed{\pi_i}, x_1 \rangle \sim x_i$$

Positive Linear Transformation:

$$u_i = a + b\pi_i, \quad b > 0, \quad i = 1, \ldots, n$$

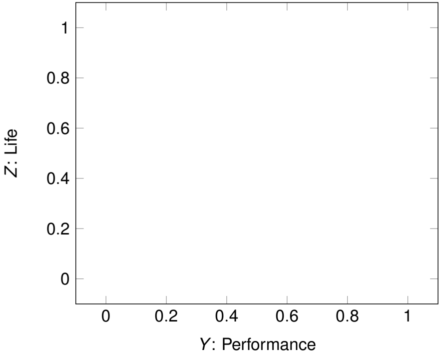
Qualitative Characteristics of Utility: Monotonicity, Risk aversion, Increasing, decreasing, and constant risk aversion

From Single-attribute Utility to Multi-attibute Utility

Assume that the attributes for the problem are $X, Y, Z, \ldots$
x denotes a specific level of X; y denotes a specific level of Y, etc.

Goal: to find a utility function over the attribute space

$$u() = u(x, y, z, \ldots)$$



Question

How can we adapt the procedure for single attribute utility functions to multiple attributes?

Module Overview

1. Approaches for Multiattribute Assessment
   - Direct Utility Assessment
   - Conditional Assessments
   - Assessing Utility Functions over “Value” Functions
   - Qualitative Structuring of Preferences
2. Utility Independence
   - Conditional Utility Independence
   - Mutual Utility Independence
   - Additive Independence and Additive Utility Function
   - General Case – No Utility Independence
3. Multiattribute Utility Theory: More than Two Attributes
   - Utility Independence
   - Mutual Utility Independence
   - Additive Utility Function
4. Multiattribute Assessment Procedure
   - Multiattribute Assessment - Steps
   - Example of Multiattribute Utility Assessment

Approaches for Multiattribute Assessment

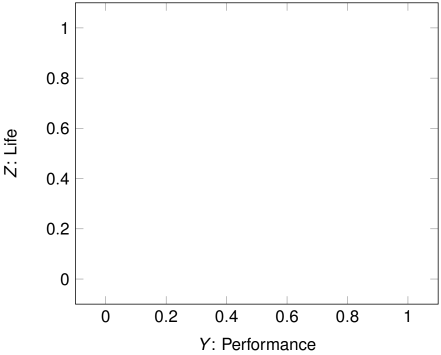
Potential Approaches

1. Direct utility assessment
2. Fixing all attributes except for one
3. Using value functions
4. Qualitative structuring of preferences

1. Direct Utility Assessment

Assign utilities to possible consequences (y_i, z_j) directly.
Arbitrarily set $u(y^o, z^o) = 0$ and $u(y^*, z^*) = 1$ where ^{o} is the least preferred and * is the most preferred.

If lottery $\langle (y^*, z^*), \boxed{\pi}, (y^o, z^o) \rangle \sim (y_i, z_j)$ then $u(y_i, z_j) = \pi$



1. Direct Utility Assessment

Limitations

Shortcomings of the procedure:

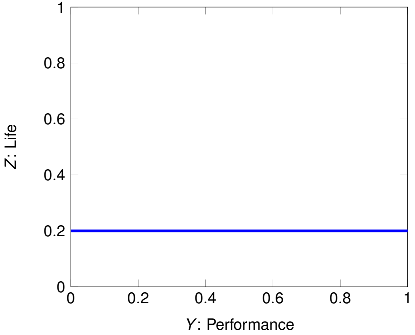
1. it fails to exploit the basic preference structure of the decision maker,
2. the requisite information is difficult to assess, and
3. the result is difficult to work with in expected utility calculations and sensitivity analysis.

2. Fixing all Attributes Except One

Consider a two attribute scenario with consequence space defined by Y, Z. Fix $Z = z_f$ and carry out utility assessments for single attribute Y.

$$y_i \sim \langle y_n, \boxed{\pi_i}, y_1 \rangle$$

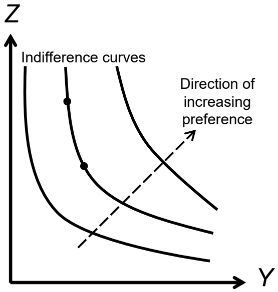
$\pi_i(y)$ is the *conditional utility function* for y values, conditioned on $Z = z_f$, and normalized by $\pi_i(y_1) = 0$ and $\pi_i(y_n) = 1$.



3. Using Value Functions

A utility function is a value function, but a value function is not a utility function!

1. Assign a value function for each point (y, z) in the consequence space.
2. The utility function is monotonically increasing in v.
3. Assess unidimensional utility functions for $u[v(y, z)]$



3. Using Value Functions

Example

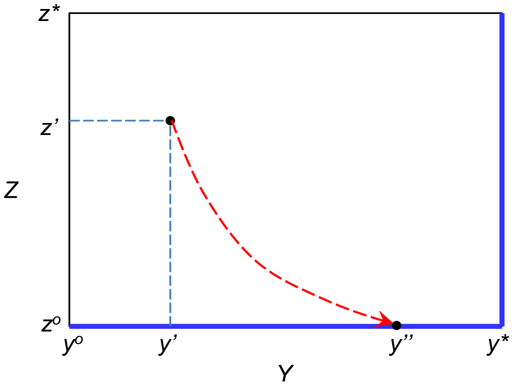


Figure: 5.1 on page 221 (Keeney and Raiffa) [adapted]

4. Qualitative Structuring of Preferences

Basic approach:

1. Postulate various assumptions about the preference attitudes of the decision maker.
2. Derive functional forms of the multiattribute utility function consistent with these assumptions.

Utility Independence

Utility Independence

Important property: Independence

Ideal scenario: Utility function such that

$$u(x, y, z, \ldots) = f[f_1(x), f_2(y), f_3(z), \ldots]$$

Conditional Utility Independence

Definition (Utility Independence)

Y is utility independent of Z when conditional preferences for lotteries on Y given z do not depend on the particular level of z.

What is the certainty equivalent $(\hat{y}, z^o)$ for the lottery $\langle (y^*, z^o), 0.5, (y^o, z^o) \rangle$? (Note: z is fixed at z^o)

Does $\hat{y}$ change if z is changed to another value (say z')?

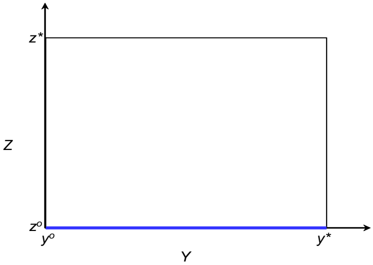


Figure: 5.2 on page 225 (Keeney and Raiffa)

Conditional Utility Independence

Question

If Y is utility independent of Z, does that also mean that Z is utility independent of Y?

Conditional Utility Independence

If Y is utility independent of Z, all conditional utility functions along horizontal cuts would be positive linear transformations of each other.

Therefore,

$$u(y, z) = g(z) + h(z)u(y, z')$$

for an arbitrarily chosen z'. In other words, the conditional utility function over Y given z does not **strategically** depend on z.

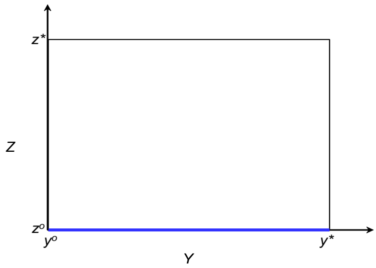


Figure: 5.2 on page 225 (Keeney and Raiffa)

Some Examples

Which utility functions satisfy the conditional utility independence conditions? If so, for which attribute?

1. $u(y,z) = \dfrac{y^\alpha z^\beta}{y+z}$
2. $u(y,z) = g(z) + h(z)u_Y(y)$
3. $u(y,z) = k(y) + m(y)u_Z(z)$
4. $u(y,z) = k_1 u_Y(y) + k_2 u_Z(z) + k_3 u_Y(y)u_Z(z)$
5. $u(y,z) = [\alpha + \beta u_Y(y)][\gamma + \delta u_Z(z)]$
6. $u(y,z) = k_Y u_Y(y) + k_Z u_Z(z)$

Assessing Utility Functions with One Utility-Independent Attribute

Assuming that Z is utility independent of Y, for any arbitrary y_0,

$$u(y, z) = c_1(y) + c_2(y)u(y_0, z), \qquad c_2(y) > 0, \qquad \forall y$$

The two-attribute utility function can be specified by:

1. three conditional utility functions, or
2. two conditional utility functions and an isopreference curve, or
3. one conditional utility function and two isopreference curve

1. Assessment using Three Conditional Utility Functions

If Z is utility independent of Y, then

$$u(y,z) = u(y,z_0)[1-u(y_0,z)]+u(y,z_1)u(y_0,z)$$

where $u(y,z)$ is normalized by
$u(y_0,z_0) = 0$ and $u(y_0,z_1) = 1$

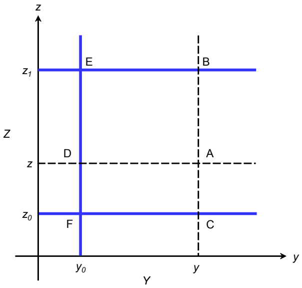


Figure: 5.7 on page 244 (Keeney and Raiffa)

2. Assessment using Two Conditional Utility Functions and One Isopreference Curve

Theorem

If Z is utility independent of Y, then

$$u(y,z) = u(y,z_0) + \left[\frac{u(y_0,z_1) - u(y,z_0)}{u(y_0,z_n(y))}\right] u(y_0,z)$$

where $u(y_0,z_0) = 0$, and $z_n(y)$ is defined such that $(y, z_n(y)) \sim (y_0, z_1)$ for an arbitrary z_1.

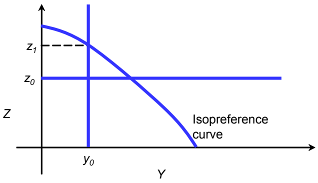


Figure: 5.9 on page 247 (Keeney and Raiffa)

3. Assessment using One Conditional Utility Functions and Two Isopreference Curves

Theorem

If Z is utility independent of Y, then

$$u(y,z) = \frac{u(y_0,z) - u(y_0,z_m(y)))}{u(y_0,z_n(y)) - u(y_0,z_m(y))}$$

where

1. $u(y,z)$ is normalized by $u(y_0,z_0) = 0$ and $u(y_0,z_1) = 1$
2. $z_m(y)$ is defined such that $(y,z_m(y)) \sim (y_0,z_0)$
3. $z_n(y)$ is defined such that $(y,z_n(y)) \sim (y_0,z_1)$

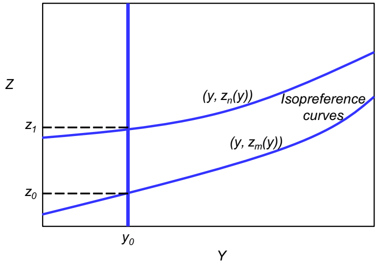


Figure: 5.11 on page 250 (Keeney and Raiffa)

Mutual Utility Independence

Stricter Condition than Conditional Utility Independence

For mutual utility independence of Y and Z,

1. Y must be utility independent of Z, i.e.,

$$u(y,z) = c_1(z) + c_2(z)u(y,z') \qquad \forall y,z$$

   for an arbitrarily chosen z', and

2. Z must be utility independent of Y, i.e.,

$$u(y,z) = d_1(y) + d_2(y)u(y',z) \qquad \forall y,z$$

   for an arbitrarily chosen y'.

Mutual Utility Independence

The Multilinear Utility Function

When Y and Z are mutually utility independent, then $u(y, z)$ can be expressed by the **multilinear representation**:

$$u(y, z) = k_Y u_Y(y) + k_Z u_Z(z) + k_{YZ} u_Y(y) u_Z(z)$$

where $k_Y > 0$, and $k_Z > 0$.

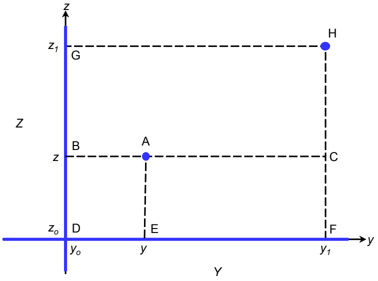


Figure: 5.4 on page 233 (Keeney and Raiffa)

Special Case of Multilinear Utility Function – Two Attributes

Theorem

If Y and Z are mutually utility independent, then the two-attribute utility function is multilinear. In particular, u can be written in the form
$u(y,z) = k_Y u_Y(y) + k_Z u_Z(z) + k_{YZ} u_Y(y) u_Z(z)$, *or*
$u(y,z) = u(y,z_0) + u(y_0,z) + k u(y,z_0) u(y_0,z)$, *where*

1. $u(y,z)$ *is normalized by* $u(y_0,z_0) = 0$ *and* $u(y_1,z_1) = 1$ *for arbitrary* y_1 *and* z_1 *such that* $(y_1,z_0) \succ (y_0,z_0)$ *and* $(y_0,z_1) \succ (y_0,z_0)$
2. $u_Y(y)$ *is conditional utility on* Y *normalized by* $u_Y(y_0) = 0$, $u_Y(y_1) = 1$
3. $u_Z(z)$ *is conditional utility on* Z *normalized by* $u_Z(z_0) = 0$, $u_Z(Z_1) = 1$
4. $k_Y = u(y_1,z_0)$, $k_Z = u(y_0,z_1)$, $k_{YZ} = 1 - k_Y - k_Z$, *and* $k = \frac{k_Y Z}{k_Y k_Z}$

Use of Isopreference Curves

Theorem

If Y and Z are mutually utility independent, then

$$u(y,z) = \frac{u(y_0,z) - u(y_0,z_n(y))}{1 + ku(y_0,z_n(y))}$$

where

1. $u(y_0,z_0) = 0$
2. $z_n(y)$ *is defined such that* $(y, z_n(y)) \sim (y_0, z_0)$
3. $k = \dfrac{u(y_0,z_1) - u(y_1,z_1) - u(y_0,z_n(y_1))}{u(y_1,z_1)u(y_0,z_n(y_1))}$ *where* (y_1, z_1) *is arbitrarily chosen such that* (y_0, z_0) *and* (y_1, z_1) *are not indifferent.*

The Multiplicative Representation

If two attributes are mutually utility independent, their utility function can be represented by either a *product form*, when $k \neq 0$, or an *additive form*, when $k = 0$.

The multilinear form

$$u(y, z) = u(y, z_0) + u(y_0, z) + ku(y, z_0)u(y_0, z)$$

Let

$$\begin{aligned} u'(y, z) &= ku(y, z) + 1 \\ &= ku(y_0, z) + ku(y, z_0) + k^2 u(y_0, z)u(y, z_0) + 1 \\ &= [ku(y, z_0) + 1][ku(y_0, z) + 1] \\ &= u'(y, z_0)u'(y_0, z) \end{aligned}$$

i.e., the product form!

Additive Utility Function

A Special Case of Conditional Mutual Independence

For $k = 0$, the utility function reduces to an additive function.

Additive utility function:

$$u(y, z) = k_Y u_Y(y) + k_Z u_Z(z)$$

where k_Y and k_Z are positive scaling constants.

Additive utility function implies that Y and Z are **mutually utility independent**. But the converse is *not true*.

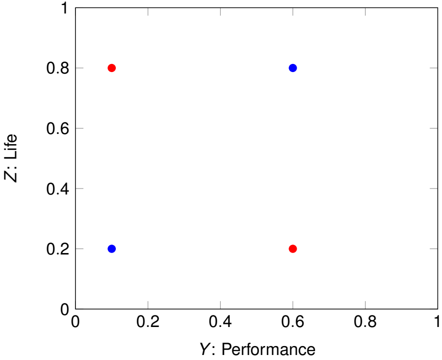
For example, $u(y, z) = y^{\alpha} z^{\beta}, 1 \leq y \leq 10, 1 \leq z \leq 10$

Checking for Additive Independence

For additive independence, the following two lotteries must be equally preferable:

$$\langle (y, z), 0.5, (y', z') \rangle \sim \langle (y, z'), 0.5, (y', z) \rangle$$

for all (y, z) given arbitrarily chosen y' and z'.



Fundamental Result of Additive Utility Function

Theorem

Attributes Y and Z are additive independent if and only if the two-attribute utility function is additive. The additive form may be either $u(y,z) = k_Y u_Y(y) + k_Z u_Z(z)$ or $u(y,z) = u(y,z^o) + u(y^o,z)$, where

1. *$u(y,z)$ is normalized by $u(y^o,z^o) = 0$ and $u(y^1,z^1) = 1$ for arbitrary y^1 and z^1 such that $(y^1,z^o) \succ (y^o,z^o)$ and $(y^o,z^1) \succ (y^o,z^o)$*
2. *$u_Y(y)$ is a conditional utility function on Y normalized by $u_Y(y^o) = 0$ and $u_Y(y^1) = 1$*
3. *$u_Z(z)$ is a conditional utility function on Z normalized by $u_Z(z^o) = 0$ and $u_Z(z^1) = 1$*
4. *$k_Y = u(y^1,z^o)$ and $k_Z = u(y^o,z^1)$*

Interpretation of Parameter k

$$\left[\langle A, 0.5, C\rangle \left\{ \begin{array}{c} \succ \\ \sim \\ \prec \end{array} \right\} \langle B, 0.5, D\rangle\right] \Leftrightarrow k \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0 \leftrightarrow \left\{ \begin{array}{l} Y \text{ and } Z \text{ are complements} \\ \text{no interaction of preference} \\ Y \text{ and } Z \text{ are substitutes} \end{array} \right.$$

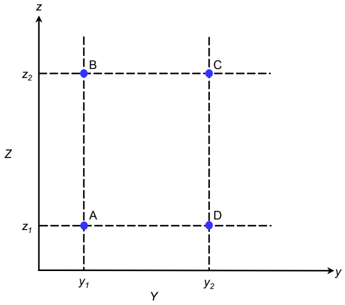


Figure: 5.6 on page 240 (Keeney and Raiffa)

Benefits of Utility Independence in Utility Assessment

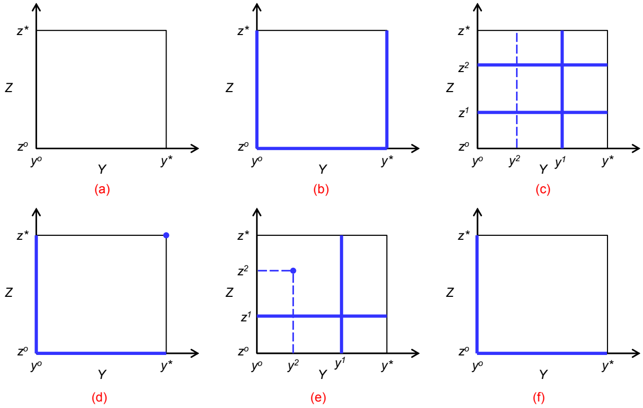


Figure: 5.3 on page 228 (Keeney and Raiffa) (a) No independence condition holds, (b) Y is utility independent of Z, (c) Z is utility independent of Y, (d, e) Y, Z are mutually utility independent, (f) Additivity assumption holds.

Degrees of Freedom for Assigning Utility Functions

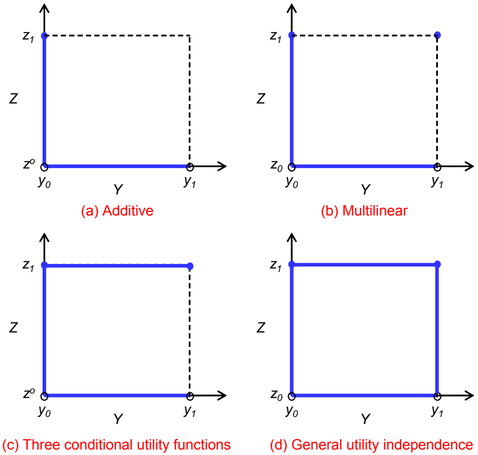


Figure: 5.12 on page 253 (Keeney and Raiffa)

General Case – No Utility Independence

Possibilities:

1. Transformation of Y and Z to new attributes that might allow exploitation of utility independence properties
2. Direct assessment of $u(y, z)$ for discrete points and then interpolation/curve fitting.
3. Application of independence results in subsets of the $Y \times Z$ space.
4. Development of more complicated assumptions about the preference structure that imply more general utility functions.

Multiattribute Utility Theory: More than Two Attributes

Utility Independence

Two Attributes

Definition (Utility Independence)

Y is utility independent of *Z* when conditional preferences for lotteries on *Y* given *z* do not depend on the particular level of *z*.

If *Y* is utility independent of *Z*, all conditional utility functions along horizontal cuts would be positive linear transformations of each other.

Therefore,

$$u(y, z) = g(z) + h(z)u(y, z')$$

for an arbitrarily chosen z'.

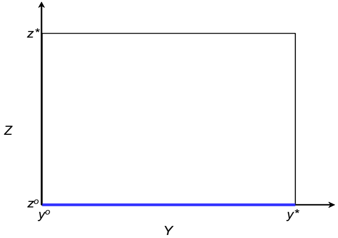


Figure: 5.2 on page 225 (Keeney and Raiffa)

Preference Independence

Generalization

Definition (Preference Independence (PI))

A subset S of attributes is **preferentially independent** of its complement $\bar{S}$ if the preference order of the **consequences** involving only changes in levels of S does not depend on the levels at which attributes in $\bar{S}$ are held fixed.

Example: Attribute set: $\{X, Y, Z, W\}$

$$S = \{X, Y\}$$

$$\bar{S} = \{Z, W\}$$

Preferential independence implies that the conditional indifference curves over S do not depend on attributes in $\bar{S}$.

Utility Independence

Generalization

Definition (Utility Independence (UI))

A subset S of attributes is **utility independent** of its complement $\bar{S}$ if the preference order of the **lotteries** involving only changes in levels of S does not depend on the levels at which attributes in $\bar{S}$ are held fixed.

Utility independence is a stronger condition. If S is UI then S is PI. The converse is not true.

Utility Independence

Three Attributes

Result for three attributes

IF

- X is utility independent of $\{Y, Z\}$, and
- $\{X, Y\}$ is preferentially independent of Z, and
- $\{X, Z\}$ is preferentially independent of Y,

THEN

$$\begin{aligned} u(x, y, z) =& k_1 u_1(x) + k_2 u_2(y) + k_3 u_3(z) \\ & + k k_1 k_2 u_1(x) u_2(y) + k k_1 k_3 u_1(x) u_3(z) + k k_2 k_3 u_2(y) u_3(z) \\ & + k^2 k_1 k_2 k_3 u_1(x) u_2(y) u_3(z) \end{aligned}$$

Utility Independence

Three Attributes

$$
\begin{aligned}
u(x,y,z) =& k_1 u_1(x) + k_2 u_2(y) + k_3 u_3(z) \\
&+ k k_1 k_2 u_1(x) u_2(y) + k k_1 k_3 u_1(x) u_3(z) + k k_2 k_3 u_2(y) u_3(z) \\
&+ k^2 k_1 k_2 k_3 u_1(x) u_2(y) u_3(z)
\end{aligned}
$$

If $k \neq 0$, we get the **multiplicative form**:

$$u'(x,y,z) = u'_1(x) u'_2(y) u'_3(z)$$

If $k = 0$, we get the **additive form**:

$$u(x,y,z) = k_1 u_1(x) + k_2 u_2(y) + k_3 u_3(z)$$

Mutual Utility Independence

Stricter Condition than Conditional Utility Independence

For mutual utility independence of Y and Z:

- Y must be utility independent of Z
  and
- Z must be utility independent of Y

Mutual Utility Independence

Theorem

If Y and Z are mutually utility independent, then the two-attribute utility function is **multilinear**. *In particular, u can be written in the form*

$$u(y,z) = k_Y u_Y(y) + k_Z u_Z(z) + k_{YZ} u_Y(y) u_Z(z), \text{ or}$$

$$u(y,z) = u(y,z_0) + u(y_0,z) + k u(y,z_0) u(y_0,z), \text{ where}$$

1. $u(y,z)$ is normalized by $u(y_0,z_0) = 0$ and $u(y_1,z_1) = 1$ for arbitrary y_1 and z_1 such that $(y_1,z_0) \succ (y_0,z_0)$ and $(y_0,z_1) \succ (y_0,z_0)$
2. $u_Y(y)$ is conditional utility on Y normalized by $u_Y(y_0) = 0$ and $u_Y(y_1) = 1$
3. $u_Z(z)$ is conditional utility on Z normalized by $u_Z(z_0) = 0$ and $u_Z(Z_1) = 1$
4. $k_Y = u(y_1,z_0)$, $k_Z = u(y_0,z_1)$, $k_{YZ} = 1 - k_Y - k_Z$, and $k = \dfrac{k_{YZ}}{k_Y k_Z}$

Mutual Utility Independence

Generalization to $n-$attributes

Definition (Mutual Utility Independence: Generalization to $n-$attributes)

Attributes $X_1, X_2, \ldots, X_n$ are mutually utility independent if every subset of $\{X_1, X_2, \ldots, X_n\}$ is utility independent of its complement.

Multilinear Utility Function

n Attributes

Theorem

Given the set of attributes $\{X_1, X_2, \ldots, X_n\}$ *with* $n \geq 2$*, if* X_i *is utility independent of* $\bar{X}_i$, $i = 1, 2, \ldots, n$*, then*

$$\begin{aligned} u(x) = &\sum_{i=1}^{n} k_i u_i(x_i) \\ &+ \sum_{i=1}^{n} \sum_{j>i} k_{ij} u_i(x_i) u_j(x_j) \\ &+ \sum_{i=1}^{n} \sum_{j>i} \sum_{l>j} k_{ijl} u_i(x_i) u_j(x_j) u_l(x_l) \\ &+ \ldots \\ &+ k_{123\ldots n} u_1(x_1) u_2(x_2) \ldots u_n(x_n) \end{aligned}$$

Additive Utility Function

A Special Case of Conditional Mutual Independence

Additive utility function:

$$u(y, z) = k_Y u_Y(y) + k_Z u_Z(z)$$

where k_Y and k_Z are positive scaling constants.

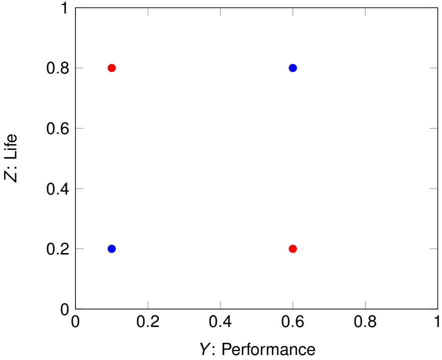
Additive utility function implies that Y and Z are **mutually utility independent**. But the converse is *not true*.

Checking for Additive Independence

For additive independence, the following two lotteries must be equally preferable:

$$\langle (y,z), 0.5, (y',z') \rangle \sim \langle (y,z'), 0.5, (y',z) \rangle$$

for all (y,z) given arbitrarily chosen y' and z'.



Additive Independence

Generalization to *n* Attributes

Definition (Additive Independence (AI))

Attributes $X_1, X_2, \ldots, X_n$ are additive independent if preferences over lotteries on $X_1, X_2, \ldots, X_n$ depend only on their marginal probability distributions and not on their joint probability distribution.

In other words, the preferences for the lotteries over $X_1 \times X_2 \times \ldots \times X_n$ can be established by comparing the values one attribute at a time.

In two-attribute case, comparing one attribute at a time in the following lotteries:

$$\langle (y, z), 0.5, (y', z') \rangle \sim \langle (y, z'), 0.5, (y', z) \rangle$$

- $E(y)$ in both lotteries is: $\dfrac{y + y'}{2}$
- $E(z)$ in both lotteries is: $\dfrac{z + z'}{2}$

Fundamental Result of Additive Utility Function

Theorem

Attributes Y and Z are additive independent if and only if the two-attribute utility function is additive. The additive form may be either $u(y,z) = k_Y u_Y(y) + k_Z u_Z(z)$ *or* $u(y,z) = u(y,z^o) + u(y^o,z)$*, where*

1. $u(y,z)$ *is normalized by* $u(y^o,z^o) = 0$ *and* $u(y^1,z^1) = 1$ *for arbitrary* y^1 *and* z^1 *such that* $(y^1,z^o) \succ (y^o,z^o)$ *and* $(y^o,z^1) \succ (y^o,z^o)$
2. $u_Y(y)$ *is a conditional utility function on Y normalized by* $u_Y(y^o) = 0$ *and* $u_Y(y^1) = 1$
3. $u_Z(z)$ *is a conditional utility function on Z normalized by* $u_Z(z^o) = 0$ *and* $u_Z(z^1) = 1$
4. $k_Y = u(y^1,z^o)$ *and* $k_Z = u(y^o,z^1)$

Additive Utility Function

For *n* Attributes

The $n-$attribute additive utility function

$$u(x) = \sum_{i=1}^{n} u(x_i, \bar{x_i^o}) = \sum_{i=1}^{n} k_i u_i(x_i)$$

is appropriate if and only if the additive independence condition holds among attributes $X_1, X_2, \ldots, X_n$, where:

1. u is normalized by $u(x_1^o, x_2^o, \ldots, x_n^o) = 0$ and $u(x_1^*, x_2^*, \ldots, x_n^*) = 1$.
2. u_i is a conditional utility function of X_i normalized by $u_i(x_i^o) = 0$ and $u_i(x_i^*) = 1, i = 1, 2, \ldots, n$.
3. $k_i = u(x_i^*, \bar{x_i^o}), i = 1, 2, \ldots, n$.

Multiattribute Assessment Procedure

Assessment Procedure for Multiattribute Utility Functions

1. Introducing the terminology and ideas.
2. Identifying relevant independence assumptions.
3. Assessing conditional utility functions or isopreference curves.
4. Assessing the scaling constants.
5. Checking for consistency and reiterating.

Fire Department Decision Making

Fundamental Objectives: Maximize the quality of fire service provided.

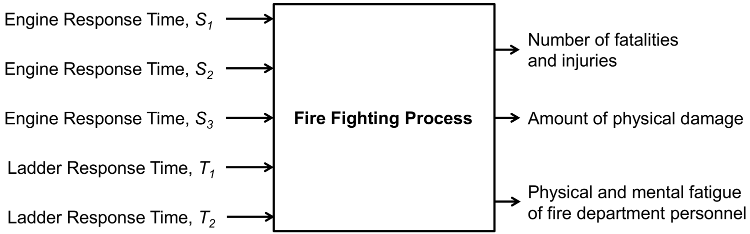
- Minimize loss of life
- Minimize injuries
- Minimize property damage
- Minimize psychological anxiety of the citizens

Alternate Policies:

- Variation in initial response pattern
- Alteration of areas of responsibility between different pieces of equipment
- Introduction of “special squads” during high demand
- Temporary relocation of equipment into other areas

Fire Department Decision Making

Figure: 7.6 on page 379 (Keeney and Raiffa)



Preference Estimation

Preferences for Alternative Courses of Action (Operational Policies)

How much is a minute of response time worth?

Attribute Space = $\{S_1, S_2, S_3, T_1, T_2\}$

A point in the attribute space = $\{3, 5, 6, 4, 7\}$ minutes.

Need to determine the response time utility function:

$$u(s_1, s_2, s_3, t_1, t_2)$$

Note that these are proxy attributes of the fundamental objectives.

Identifying Relevant Independence Assumptions

Is it reasonable to assume:

1. Engine response times $\{S_1, S_2, S_3\}$ and the ladder response times $\{T_1, T_2\}$ are utility independent of each other?
2. The j^{th} engine response S_j is utility independent of the other engine responses, for $j = 1, 2, 3$?
3. First ladder response time T_1 and the second ladder response time T_2 are utility independent of each other?

Checking for Independence

Example

Check: First ladder response time T_1 and the second ladder response time T_2 are utility independent of each other.

Question 1

If the response time of the second ladder (t_2) is fixed at **6 minutes**, what response time (t_1) for the first ladder would be indifferent to having a 50-50 chance that the first ladder responds in either 1 or 5 minutes?

$$\langle 1, 0.5, 5 \rangle \sim \boxed{t_1}$$

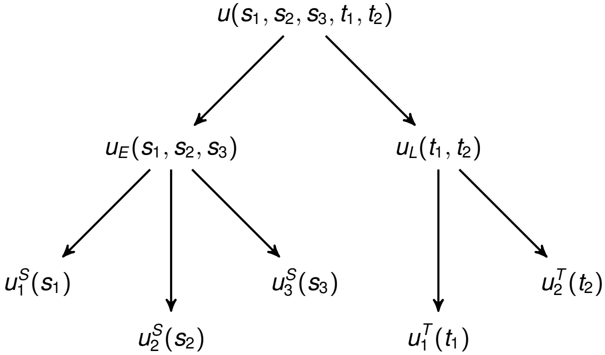
Question 2

If the response time of the second ladder (t_2) is fixed at **8 minutes**, what response time (t_1) for the first ladder would be indifferent to having a 50-50 chance that the first ladder responds in either 1 or 5 minutes?

$$\langle 1, 0.5, 5 \rangle \sim \boxed{t_1}$$

Implications of Independence Assumptions

If these independence assumptions are satisfied, then



Assessing Conditional Utility Functions

Estimating the conditional Utility function of first arriving ladder, $u_1^T(t_1)$:

1. Arbitrarily set $u_1^T(0) = 0, u_1^T(20) = -1$.
2. By mapping qualitative characteristics of this utility function, it was found that the utility function must take the following form:

$$u_1^T(t) = d + b(-e^{ct})$$

   where $d, b,$ and $c > 0$.
3. Estimate the unknowns using lottery question: $\langle 1, 0.5, 7 \rangle \sim \boxed{t_1}$. Say the response is $t_1 = 4.5$. Then,

$$u_1^T(4.5) = 0.5u_1^T(1) + 0.5u_1^T(7)$$

4. Solve the three equations for three unknowns to give:

$$u_1^T(t_1) = 0.0998(1 - e^{0.12t_1})$$

Evaluating Scaling Constants

Using the multilinear utility function,

$$u_L(t_1, t_2) = k_1 u_1^T(t_1) + k_2 u_2^T(t_2) + [k_1 + k_2 - 1] u_1^T(t_1) u_2^T(t_2)$$

Question 1

What is the response time t_2 for which you would be indifferent between $(t_1 = 3, t_2 = 8)$ and $(t_1 = 4, \boxed{t_2 = ?})$. Say the response is 5.7. Then,

$$u_L(3, 8) = u_L(4, 5.7)$$

Question 2

What is the response time t_2 for which you would be indifferent between $(t_1 = 2, t_2 = 6)$ and $(t_1 = 3, \boxed{t_2 = ?})$. Say the response is 4.2. Then,

$$u_L(2, 6) = u_L(3, 4.2)$$

Based on the responses, solve two equations with two unknowns (k_1, k_2). In this case, $k_1 = 0.66$, $k_2 = 0.19$.

The Overall Utility Function

Based on the assessments, it was found that:

$$u(s_1, s_2, s_3, t_1, t_2) = 0.24u_L(\mathbf{t}) + 0.16u_E(\mathbf{s}) - 0.6u_L(\mathbf{t})u_E(\mathbf{s})$$

$$u_E(s_1, s_2, s_3)$$

$$u_L(t_1, t_2) = 0.66u_1^T(t_1) + 0.19u_2^T(t_2) - 0.15u_1^T(t_1)u_2^T(t_2)$$

$$u_1^S(s_1) \quad u_2^S(s_2) \quad u_3^S(s_3)$$

$$u_2^T(t_2)$$

$$u_1^T(t_1) = 0.998[1 - e^{0.12t_1}]$$

Insights from the Utility Function

1. u is decreasing in each t_i and s_j.
2. Each minute of delay of the first ladder (engine) is more important* than the corresponding delay of the second ladder (engine). [Observed from the relative values of the coefficients.]
3. The conditional utility function for each attribute is risk averse regardless of the values of other attributes.
4. The relative importance of the response time of a ladder (engine) increases as the response time of the other ladder (engine) increases. (Slower the first ladder, the more important it is that the second ladder arrive soon after).

$$u_L(t_1, t_2) = 0.66u_1^T(t_1) + 0.19u_2^T(t_2) - 0.15u_1^T(t_1)u_2^T(t_2)$$

*Important means that we are willing to pay more to make the more important change.

Insights from the Overall Utility Function

$$
\begin{aligned}
u(\mathbf{s},\mathbf{t}) =& 0.24u_L(\mathbf{t}) + 0.16u_E(\mathbf{s}) - 0.6u_L(\mathbf{t})u_E(\mathbf{s}) \\
u_L(\mathbf{t}) =& 0.66u_1^T(t_1) + 0.19u_2^T(t_2) - 0.15u_1^T(t_1)u_2^T(t_2) \\
u_E(\mathbf{s}) =& 0.63u_1^S(s_1) + 0.18u_2^S(s_2) + 0.09u_3^S(s_3) - 0.06u_1^S(s_1)u_2^S(s_2) \\
& - 0.03u_1^S(s_1)u_3^S(s_3) - 0.01u_2^S(s_2)u_3^S(s_3)
\end{aligned}
$$

A one-minute delay in the arrival of the i^{th} ladder is more important than the corresponding one minute delay on the i^{th} engine.

Implication: We would prefer to have the first ladder respond in two minutes and the first engine in three minutes ($t_1 = 2, s_1 = 3$), than to have the first engine respond in two minutes and the first ladder in three ($t_1 = 3, s_1 = 2$).

Insights from the Overall Utility Function

$$
\begin{aligned}
u(\mathbf{s}, \mathbf{t}) =& 0.24 u_L(\mathbf{t}) + 0.16 u_E(\mathbf{s}) - 0.6 u_L(\mathbf{t}) u_E(\mathbf{s}) \\
u_L(\mathbf{t}) =& 0.66 u_1^T(t_1) + 0.19 u_2^T(t_2) - 0.15 u_1^T(t_1) u_2^T(t_2) \\
u_E(\mathbf{s}) =& 0.63 u_1^S(s_1) + 0.18 u_2^S(s_2) + 0.09 u_3^S(s_3) - 0.06 u_1^S(s_1) u_2^S(s_2) \\
& - 0.03 u_1^S(s_1) u_3^S(s_3) - 0.01 u_2^S(s_2) u_3^S(s_3)
\end{aligned}
$$

The relative importance of the response times of ladders increases as the response times of engines increase.

Implication: The importance of the first arriving engine is less when a ladder has already arrived than when no ladders have arrived.

Summary

Reference

1. Keeney, R. L. and H. Raiffa (1993). *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. Cambridge, UK, Cambridge University Press. Chapters 5 and 6.