Module 06
Single-Attribute Utility Theory

Dr. Jitesh H. Panchal

Decision Making in Engineering Design

School of Mechanical Engineering
Purdue University, West Lafayette, IN

Learning Objectives

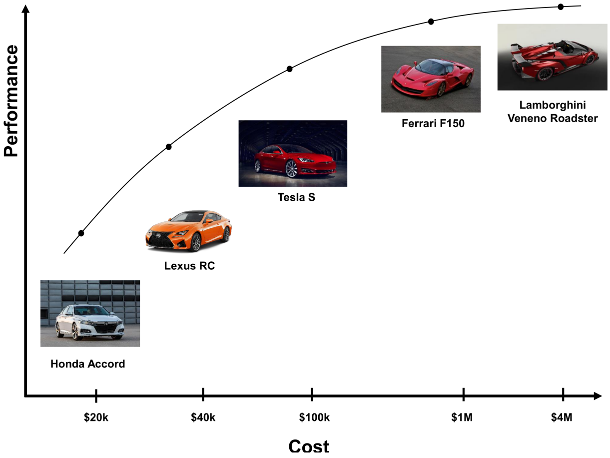
Learning Objectives for Module 06:

1. Quantifying preferences using Utility functions
2. Qualitative and quantitative characteristics of Utility functions
3. Procedures for assessing Utility functions
4. Process for making decisions using Utility theory

	Single Attribute	Multiple Attributes
Certainty	I	II
Uncertainty	III	IV

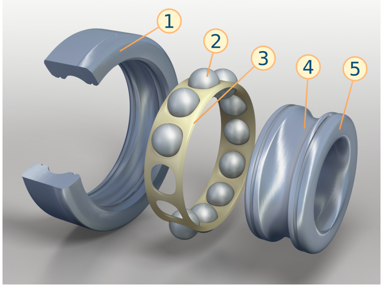
Examples

1. Decision to Buy a Car



Examples

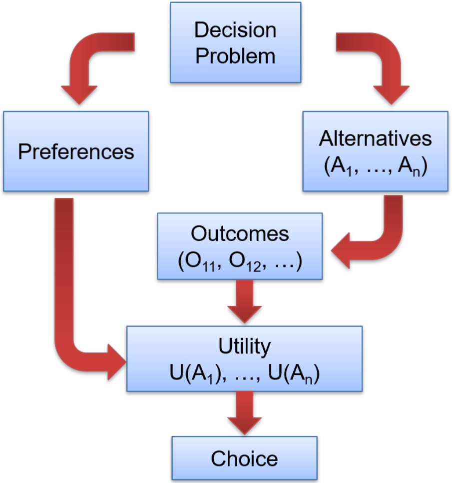
2. Rolling Element Bearings



Bearing load rating: The radial load that causes 10% of the bearings to fail at the bearing manufacturer's rating life (e.g., 1 million revolutions).

Image source: `https://commons.wikimedia.org/wiki/File:Rolling-element_bearing_(numbered).png`

The Structure of a Design Decision



Problem Statement for this Module

Problem Statement

Choose among alternatives $\{A_1, A_2, \ldots\}$, each of which will eventually result in a consequence described by one attribute X which can take values $\{x_1, x_2, \ldots\}$.

Decision maker **does not** know exactly what consequence (x_i) will result from each alternative.

But he/she **can** assign probabilities to the various consequences that might result from any alternative.

Module Overview

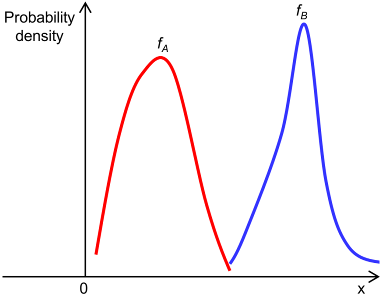
1. Decision Making under Uncertainty
   - Alternate Approaches to Risky Choice Problem
   - Motivation for Using Utility Theory
2. Unidimensional Utility Theory
   - Fundamentals of Utility Theory
   - Qualitative Characteristics of Utility
3. Attitudes to Risk
   - Risk Averse and Risk Prone
   - Restricting the form of Utility Functions
   - Measuring Risk Aversion
4. Procedures for Assessing Utility Functions

Keeney, R. L. and H. Raiffa (1993). *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. Cambridge, UK, Cambridge University Press. Chapter 4.

Decision Making under Uncertainty

Alternate Approaches to Risky Choice Problem

(a) Probabilistic dominance



Is $A \prec B$ or $B \prec A$?

Alternate Approaches to Risky Choice Problem

(a) Probabilistic dominance (contd.)

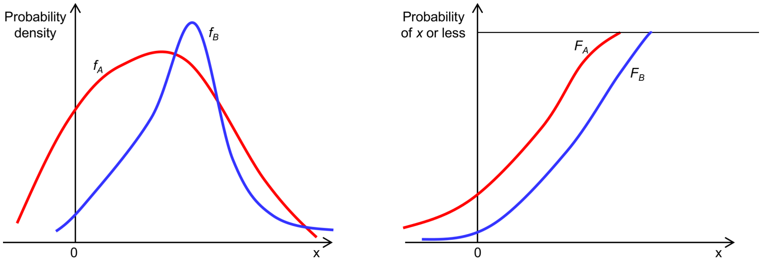
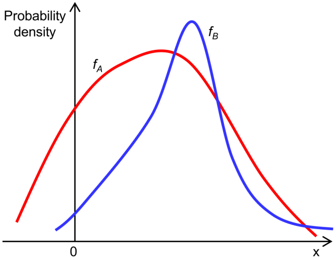


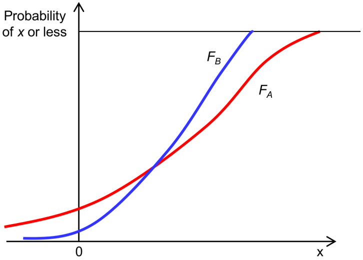
Figure: 4.2 on page 135 (Keeney and Raiffa)

Is $A \prec B$ or $B \prec A$?

Alternate Approaches to Risky Choice Problem

(a) Probabilistic dominance (contd.)





Is $A \prec B$ or $B \prec A$?

Alternate Approaches to Risky Choice Problem:

(b) Expected "**value**" of uncertain outcome

Consider the following alternatives

- A_1: Earn \$100,000 for sure
- A_2: Earn \$200,000 or \$0, each with probability 0.5
- A_3: Earn \$1,000,000 with probability 0.1 or \$0 with probability 0.9
- A_4: Earn \$200,000 with probability 0.9 or lose \$800,000 with probability 0.1

The expected amount earned is exactly \$100,000 in each alternative.

Are all alternatives equally desirable?

Alternate Approaches to Risky Choice Problem

(c) Consideration of mean and variance

One possibility is to consider variance, in addition to the expected value of the outcome.

But, Alternatives A_3 and A_4 have the same mean and variance:

- A_3: Earn \$1,000,000 with probability 0.1 or \$0 with probability 0.9
- A_4: Earn \$200,000 with probability 0.9 or lose \$800,000 with probability 0.1

Are A_3 and A_4 equally preferred?

Alternate Approaches to Risky Choice Problem

(c) Consideration of mean and variance (contd.)

Limitations:

- Any measure that considers mean and variance only cannot distinguish between these two alternatives.
- Considering mean and variance imposes additional problem of finding relative preference between them.

Primary Motivation for using Utility Theory

IF an appropriate utility is assigned to each possible consequence,
AND the expected utility of each alternative is calculated,

THEN the best course of action is the alternative with the highest *expected utility*.

Unidimensional Utility Theory

Fundamentals of Utility Theory

Assume n consequences labeled $x_1, x_2, \ldots, x_n$ such that x_i is less preferred than x_{i+1}

$$x_1 \prec x_2 \prec x_3 \prec \cdots \prec x_n$$

These need not be numerical values.

The decision maker is asked to state preferences about two alternatives a' and a'', where

1. Alternative a' will result in consequence x_i with probability p'_i for $i = 1..n$
2. Alternative a'' will result in consequence x_i with probability p''_i for $i = 1..n$

Fundamentals of Utility Theory (contd.)

Assume that for each i, the decision maker is indifferent between the following options:

1. Certainty Option:

Receive x_i

2. Risky Option:

Receive x_n (best outcome) with probability π_i and x_1 (worst outcome) with probability $(1 - \pi_i)$. This option is denoted as $\langle x_n, \pi_i, x_1 \rangle$

Fundamentals of Utility Theory (contd.)

π_i's can be thought of as numerical scaling of x's. The preference structure

$$x_1 \prec x_2 \prec x_3 \prec \cdots \prec x_n$$

maps to

$$\pi_1 < \pi_2 < \pi_3 < \cdots < \pi_n$$

Clearly,

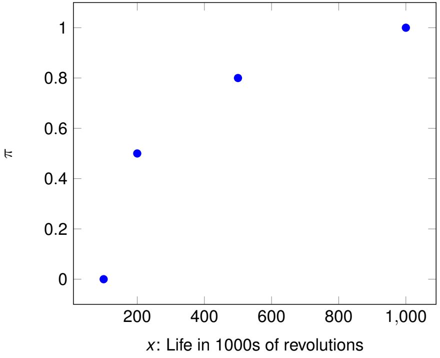
$$\pi_1 = 0$$
$$\pi_n = 1$$
$$\pi_1 < \pi_2 < \pi_3 < \cdots < \pi_n$$

Fundamentals of Utility Theory (contd.)

Example

Preferences:

- $x_1 = 100,000; \pi_1 = 0$
- $x_2 = 200,000; \pi_2 = 0.5$
- $x_3 = 500,000; \pi_3 = 0.8$
- $x_4 = 1,000,000; \pi_4 = 1$



Fundamentals of Utility Theory (contd.)

Fundamental Result of Utility Theory

The expected value of the π's can be used to numerically scale probability distributions over the x's.

The decision maker is to choose among probabilistic alternatives a' and a''

- a' : results in x_i with probability p'_i
  $\equiv \langle x_n, \bar{\pi}', x_1 \rangle$, i.e., $\bar{\pi}'$ chance at x_n and $(1 - \bar{\pi}')$ chance at x_1
- a'' : results in x_i with probability p''_i
  $\equiv \langle x_n, \bar{\pi}'', x_1 \rangle$ i.e., $\bar{\pi}''$ chance at x_n and $(1 - \bar{\pi}'')$ chance at x_1

The expected π scores for alternatives a' and a'' are as follows:

$$\bar{\pi}' = \sum_i p'_i \pi_i \qquad \text{and} \qquad \bar{\pi}'' = \sum_i p''_i \pi_i$$

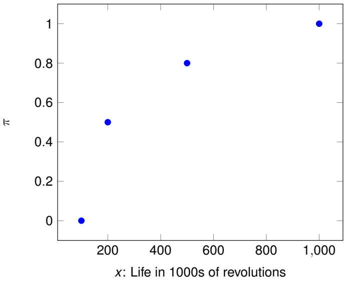
Now, we can rank order a', a'' in terms of $\bar{\pi}', \bar{\pi}''$

Fundamentals of Utility Theory (contd.)

Example

Preferences:

- $x_1 = 100,000; \pi_1 = 0$
- $x_2 = 200,000; \pi_2 = 0.5$
- $x_3 = 500,000; \pi_3 = 0.8$
- $x_4 = 1,000,000; \pi_4 = 1$



Decision: Choose among the following materials (a' and a''):

	p_1	p_2	p_3	p_4
Material a'	$p'_1 = 0.25$	$p'_2 = 0.25$	$p'_3 = 0.25$	$p'_4 = 0.25$
Material a''	$p''_1 = 0.2$	$p''_2 = 0.3$	$p''_3 = 0.3$	$p''_4 = 0.2$

Fundamentals of Utility Theory (contd.)

Transforming π's into u's using a positive linear transformation

$$u_i = a + b\pi_i, \quad b > 0, \quad i = 1, \ldots, n$$

Then,

$$u_1 < u_2 < \cdots < u_n$$

The expected u values rank order a' and a'' in the same way as the expected π values

$$\bar{u}' = \sum_i p'_i u_i = \sum_i p'_i (a + b\pi_i) = a + b\bar{\pi}'$$

Question

Does any monotonic transformation (e.g., other than positive linear) preserve the preferences to probabilistic alternatives?

Fundamentals of Utility Theory (contd.)

Essence of the problem

How can appropriate π values be assessed in a responsible manner?

Direct Assessment of Utilities

Define:

- x^o as a least preferred consequence
- x^* as a most preferred consequence.

Assign (origin and unit of measurement):

$$u(x^*) = 1 \quad and \quad u(x^o) = 0$$

Evaluate π: For each other consequence x, assign a probability π such that x is indifferent to the lottery $\langle x^*, \pi, x^o \rangle$.

Note that the expected utility of the lottery is:

$$u(x) = \pi u(x^*) + (1 - \pi) u(x^o) = \pi$$

Continue for all x's (or fit a curve).

Qualitative Characteristics of Utility

The shape and functional form of the utility function tells us a great deal about the basic attitudes of the decision maker towards risk.

1. Monotonicity
2. Certainty equivalence
3. Strategic equivalence

Qualitative Characteristics of Utility

1. Monotonicity

Definition (Monotonicity)

For a monotonically increasing utility function

$$[x_1 > x_2] \Leftrightarrow [u(x_1) > u(x_2)]$$

For a monotonically decreasing utility function

$$[x_1 > x_2] \Leftrightarrow [u(x_1) < u(x_2)]$$

Can you think of an example where the utility is non-monotonic?

Qualitative Characteristics of Utility

2. Certainty Equivalence

Assume lottery L yields consequences $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$.

Define:
$\tilde{x}$: Uncertain consequence of lottery (i.e., random variable)
$\bar{x}$: Expected consequence

The **expected consequence** of the lottery is:

$$\bar{x} \equiv E(\tilde{x}) = \sum_{i=1}^{n} p_i x_i$$

The **expected utility** of the lottery is:

$$E[u(\tilde{x})] = \sum_{i=1}^{n} p_i u(x_i)$$

Qualitative Characteristics of Utility

2. Certainty Equivalence

Definition (Certainty equivalence)

A certainty equivalent of lottery L is the amount $\hat{x}$ such that the decision maker is indifferent between L and the amount $\hat{x}$ for certain.

$$u(\hat{x}) = E[u(\tilde{x})], \quad or \quad \hat{x} = u^{-1}Eu(\tilde{x})$$

Certainty equivalent of any lottery is unique for monotonic utility functions. For non-monotonic cases, the certainty equivalent may not be unique.

Qualitative Characteristics of Utility

2. Certainty Equivalence (continuous variables)

If x is a continuous variable, the associated uncertainty is described using a probability density function, $f(x)$. Then,

$$\bar{x} \equiv E(\tilde{x}) = \int x f(x) dx$$

The certainty equivalent $\hat{x}$ is a solution to

$$u(\hat{x}) = E[u(\tilde{x})] = \int u(x) f(x) dx$$

Qualitative Characteristics of Utility

2. Certainty Equivalence – Example

$u(x) = a - be^{-cx}$ Lottery $\langle x_1, 0.5, x_2 \rangle$

Determine:

- Expected consequence, $\bar{x}$
- Certainty equivalence, $\hat{x}$

Qualitative Characteristics of Utility

2. Certainty Equivalence – Example

$u(x) = a - be^{-cx}$

The lottery is described by the uniform probability density function:

$$f(x) = \frac{1}{x_2 - x_1}, \quad x_1 \leq x \leq x_2$$

Determine:

- Expected consequence, $\bar{x}$
- Certainty equivalence, $\hat{x}$

Qualitative Characteristics of Utility

3. Strategic Equivalence

Definition (Strategic equivalence)

Two utility functions, u_1 and u_2, are strategically equivalent ($u_1 \sim u_2$) if and only if they imply the same preference ranking for any two lotteries.

If two utility functions are strategically equivalent, the certainty equivalents of two lotteries must be the same. Therefore,

$$u_1 \sim u_2 \Rightarrow u_1^{-1} E u_1(\tilde{x}) = u_2^{-1} E u_2(\tilde{x}), \quad \forall \tilde{x}$$

Qualitative Characteristics of Utility

3. Strategic Equivalence (contd.)

For some constants h and $k > 0$, if

$$u_1(x) = h + ku_2(x), \quad \forall x$$

then $u_1 \sim u_2$

Theorem

If $u_1 \sim u_2$, there exists two constants h and $k > 0$ such that

$$u_1(x) = h + ku_2(x), \quad \forall x$$

Example: $u(x) = a + bx \sim x, b > 0$

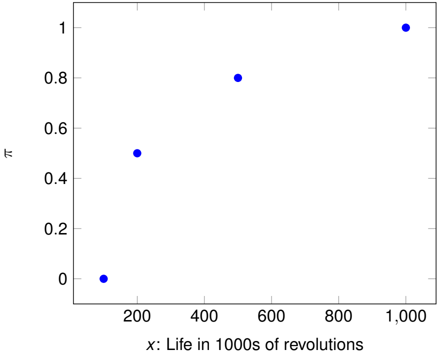
We can show that if the utility function is linear, the certainty equivalent for any lottery is equal to the expected consequence of that lottery.

Single Attribute Decision Making Under Uncertainty

An Example

Preferences:

- $x_1 = 100,000; \pi_1 = 0$
- $x_2 = 200,000; \pi_2 = 0.5$
- $x_3 = 500,000; \pi_3 = 0.8$
- $x_4 = 1,000,000; \pi_4 = 1$



Attitudes to Risk

Risk Aversion – An Illustration

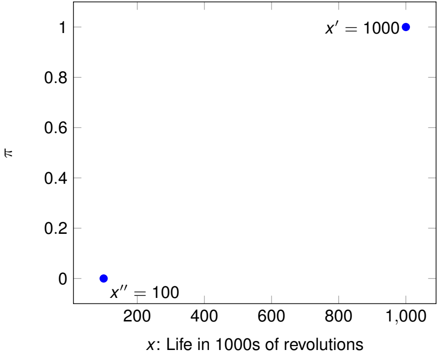
Consider a lottery $\tilde{x} = \langle x', 0.5, x'' \rangle$.

The expected consequence of the lottery is:

$$\bar{x} = E(\tilde{x}) = 0.5x' + 0.5x'' = \frac{x' + x''}{2}.$$

Choose between:

- $\bar{x}$ for certain, and
- lottery $\tilde{x} = \langle x', 0.5, x'' \rangle$



If the decision maker prefers the certain outcome $\bar{x}$, then the decision maker prefers to *avoid risks* $\Rightarrow$ Risk Averse.

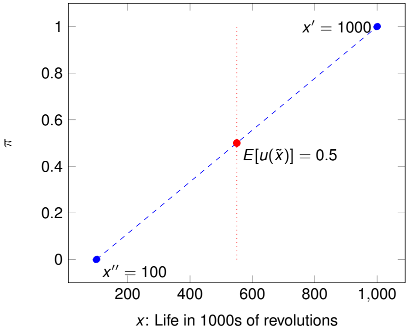
Definition of Risk Aversion

Definition (Risk Aversion)

A decision maker is risk averse if he prefers the expected consequence of any non-degenerate lottery to that lottery.

Let the possible consequences of any lottery are represented by $\tilde{x}$, a decision maker is risk averse if, for all non-degenerate lotteries, utility of expected consequence is greater than the expected utility of that lottery, i.e.,

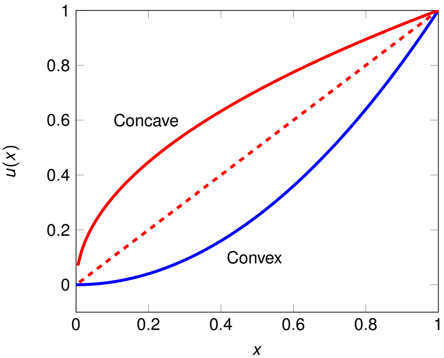
$$u[E(\tilde{x})] > E[u(\tilde{x})]$$



Risk Aversion and Utility Functions

Theorem

A decision maker is risk averse if and only if his/her utility function is concave.



Risk Prone and Risk Neutral

Definition (Risk Prone)

A decision maker is risk prone if (s)he prefers any non-degenerate lottery to the expected consequence of that lottery.

$$u[E(\tilde{x})] < E[u(\tilde{x})]$$

Definition (Risk Neutral)

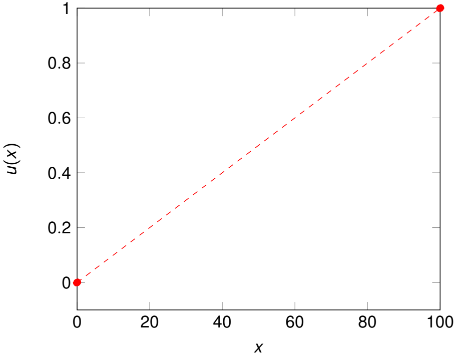
A decision maker is risk neutral if (s)he is indifferent between any non-degenerate lottery and the expected consequence of that lottery.

$$u[E(\tilde{x})] = E[u(\tilde{x})]$$

Restricting the Form of Utility Functions

Qualitative characteristics:

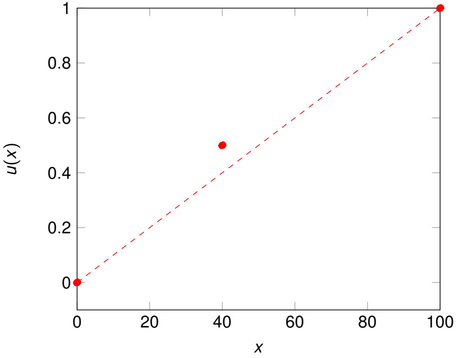
- Monotonicity
- Risk aversion



Restricting the Form of Utility Functions

Qualitative characteristics:

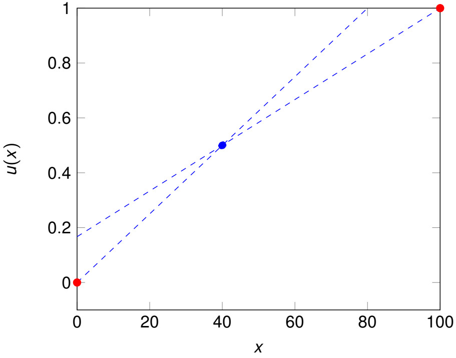
- Monotonicity
- Risk aversion



Restricting the Form of Utility Functions

Qualitative characteristics:

- Monotonicity
- Risk aversion



Risk Premium

Definition (Risk Premium of a lottery)

The **risk premium** (RP) of a lottery $\tilde{x}$ is its expected value ($\bar{x}$) minus its certainty equivalent ($\hat{x}$).

$$RP(\tilde{x}) = \bar{x} - \hat{x} = E(\tilde{x}) - u^{-1}Eu(\tilde{x})$$

The **risk premium** (RP) is the amount of the attribute that a (risk averse) decision maker is willing to "give up" from the average to avoid the risks associated with the particular lottery.

Risk Premium

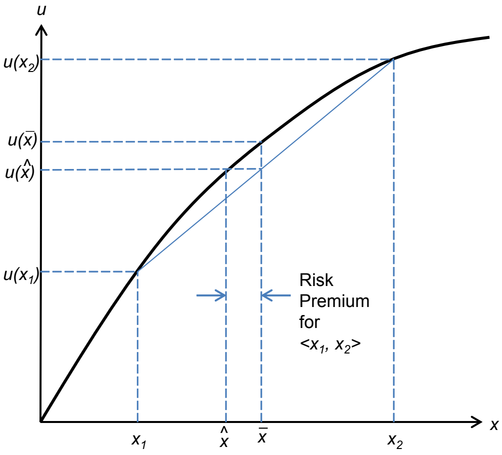


Figure: 4.5 on page 152 (Keeney and Raiffa)

Insurance Premium

The **insurance premium** (*IP*) for a lottery $\tilde{x}$ is the negative of the certainty equivalent of the lottery.

$$IP(\tilde{x}) = -\hat{x} = -u^{-1}Eu(\tilde{x})$$

The insurance premium is the amount that the decision maker is willing to give up to rid himself of the financial responsibility of the lottery.

Risk Premium

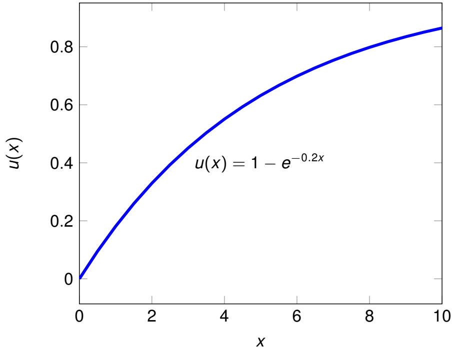
Example

For the utility function

$$u(x) = 1 - e^{-0.2x}$$

Determine the following for the lottery $\tilde{x} = \langle 10, 0.5, 0 \rangle$

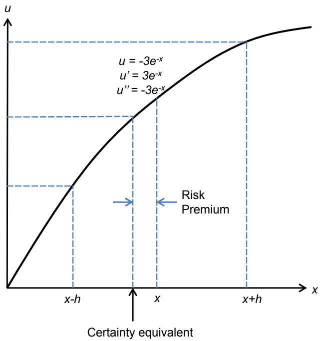
- Expected consequence ($\bar{x}$)
- Certainty equivalent ($\hat{x}$)
- Risk premium (RP)
- Insurance premium (IP)

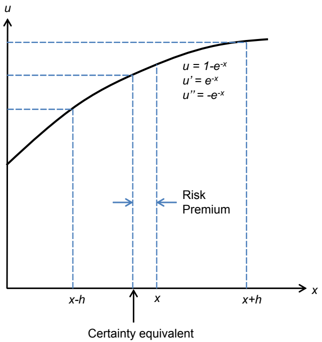


Measuring Risk Aversion

Possible Measures of Risk Aversion

- Risk premium (*RP*)
- Curvature of the utility function





A Measure of Risk Aversion

Definition (Risk aversion)

The local risk aversion at x, written $r(x)$, is defined by

$$r(x) = -\frac{u''(x)}{u'(x)}$$

- $r(x) > 0 \Rightarrow$ Risk Averse
- $r(x) < 0 \Rightarrow$ Risk Prone

Characteristics of this measure:

1. it indicates whether the utility function is risk averse or risk prone
2. shows equivalence between two strategically equivalent utility functions

A Measure of Risk Aversion

Example

Determine $r(x)$ for:

$$u(x) = a - be^{-cx}$$

Local Risk Aversion – Some Results

Theorem

Two utility functions are strategically equivalent if and only if they have the same risk-aversion function.

Theorem

If $r(x)$ is positive for all x, then u is concave and the decision maker is risk-averse.

Constant, Decreasing, and Increasing Risk Aversion

Let x denote a decision maker's endowment of a given attribute X. Now, add to x a lottery $\tilde{x}$ involving only a small range of X with an expected value of zero. Let the risk premium of this lottery be $RP_x(\tilde{x})$.

What happens to $RP_x(\tilde{x})$ as x increases?

As you become richer, are you willing to give up more (or less) to take the same amount of risk?

Constant, Decreasing, and Increasing Risk Aversion

Example 1

How does risk aversion change with x for the quadratic utility function:

$$u(x) = a + bx - cx^2$$

where $b > 0, c > 0, x < \dfrac{b}{2c}$

Is this an appropriate functional form for decreasing risk aversion?

Constant, Decreasing, and Increasing Risk Aversion

Example 2

How does risk aversion change with x for the logarithmic utility function:

$$u(x) = \log(x + b)$$

where $x > -b$

Is this an appropriate functional form for decreasing risk aversion?

Constant, Decreasing, and Increasing Risk Aversion

Implication

Many of the traditional candidates for a utility function (e.g., exponential and quadratic) are not appropriate for a decreasingly risk averse decision maker.

Theorem

The risk aversion r is constant if and only if $\pi(x, \tilde{x})$ is a constant function of x for all $\tilde{x}$.

Theorem

$$u(x) \sim -e^{-cx} \Leftrightarrow r(x) \equiv c > 0 \qquad \textit{(constant risk aversion)}$$

$$u(x) \sim x \Leftrightarrow r(x) \equiv 0 \qquad \textit{(risk neutrality)}$$

$$u(x) \sim e^{-cx} \Leftrightarrow r(x) \equiv c < 0 \qquad \textit{(constant risk proneness)}$$

Constant, Decreasing, and Increasing Risk Aversion

Example 3

How does risk aversion change with x for the following utility function:

$$u(x) = -e^{-ax} - be^{-cx}$$

where $a > 0, b > 0, c > 0$.

Determine the conditions under which this function is :

- constantly risk averse
- decreasingly risk averse
- increasingly risk averse

Common Decreasingly Risk Averse Utility Functions

Table: 4.5 on page 173 (Keeney and Raiffa)

$u(x)$	Restrictions	$r(x)$	Decreasing Risk Aversion Range
$\log(x+b)$	-	$\frac{1}{x+b}$	$x \geq -b$
$(x+b)^c$	$0 < c < 1$	$-\frac{c-1}{x+b}$	$x \geq -b$
$(x+b)^{-c}$	$c > 0$	$\frac{c+1}{x+b}$	$x \geq -b$
$x + c\log(x+b)$	$c > 0$	$\frac{c}{(x+b)(x+c+b)}$	$x > -b$
$-e^{-ax} - be^{-cx}$	$a, b, c > 0$	$\frac{a^2 e^{-ax} + bc^2 e^{-cx}}{ae^{-ax} + bce^{-cx}}$	all x
$-e^{-ax} + bx$	$a, b > 0$	$\frac{a^2 e^{-ax}}{ae^{-ax} + b}$	all x

Monotonically Decreasing Utility Functions

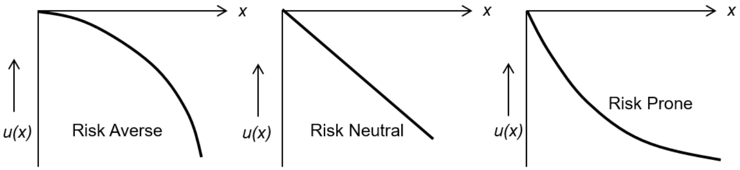


Figure: 4.15 on page 180 (Keeney and Raiffa)

A measure of risk aversion for a decreasing utility function is

$$q(x) \equiv \frac{u''(x)}{u'(x)} = \frac{d}{dx}[\log(u'(x))]$$

Note: This is same as $r(x)$, except the $-$ve sign. Positive value of $q(x)$ means that the decision maker is risk averse.

Non-monotonic Utility Functions

For non monotonic preferences, a decision maker is risk averse [risk prone] if and only if his utility function is concave [convex].

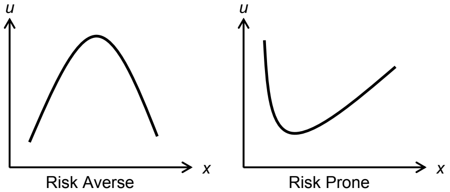


Figure: 4.18 on page 188 (Keeney and Raiffa)

For non-monotonic utility functions, the certainty equivalent is not necessarily unique. The risk premium and measure of risk aversion cannot be usefully defined.

Procedures for Assessing Utility Functions

A Generic Procedure for Assessing Utility Functions

1. Preparing for assessment
2. Identifying the relevant qualitative characteristics
3. Specifying quantitative restrictions
4. Choosing a utility function
5. Checking for consistency

1. Preparing for Assessment

- Structuring the decision problem
- Developing an appropriate scale for measuring the attribute
- Acquainting the decision maker with the framework

Educate the decision maker (not bias him/her) and hopefully force him/her to think about his/her preferences.

2. Identifying the Relevant Qualitative Characteristics

1. Determine monotonicity.
2. Determine whether the decision maker is risk averse, risk neutral, or risk prone.
3. Determine whether the decision maker is increasingly, decreasingly, or constantly risk averse.

3. Specifying Quantitative Restrictions

Fixing utilities for a few points on the utility function.
Involves determining the certainty equivalents of a few lotteries.

Example: Five point assessment procedure for utility functions:

$$u(x_{0.5}) = \frac{1}{2}u(x_1) + \frac{1}{2}u(x_0)$$
$$u(x_{0.75}) = \frac{1}{2}u(x_1) + \frac{1}{2}u(x_{0.5})$$
$$u(x_{0.25}) = \frac{1}{2}u(x_0) + \frac{1}{2}u(x_{0.5})$$

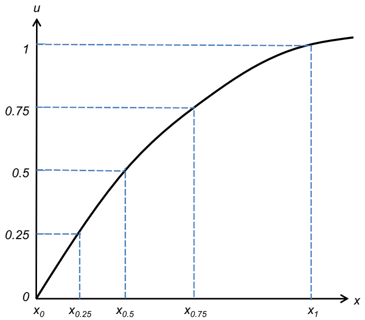


Figure: 4.22 on page 195 (Keeney and Raiffa)

3. Specifying Quantitative Restrictions

a. Standard Gamble Methods

1. Preference comparison methods:

$$\langle x', p, x'' \rangle \quad \boxed{???} \quad x$$

2. Probability equivalence methods:

$$\langle x_{i+1}, \boxed{???}, x_{i-1} \rangle \sim x_i$$

3. Value equivalence methods:

$$\langle x_{i+1}, 0.5, \boxed{x_{i-1}} \rangle \sim x_i$$

4. Certainty equivalence methods

$$\langle x_n, p, x_0 \rangle \sim \boxed{x_i}$$

3. Specifying Quantitative Restrictions

b. Paired Gamble Methods

1. Preference comparison methods:

$$\langle x', p, x'' \rangle \quad \boxed{???} \quad \langle x''', p, x'''' \rangle$$

2. Probability equivalence methods:

$$\langle x_{i+1}, 0.5, x_{i-1} \rangle \sim \langle x_{n+1}, \boxed{???}, x_0 \rangle$$

3. Value equivalence methods:

$$\langle \boxed{???}, p_1, x'' \rangle \sim \langle x''', p_2, x'''' \rangle$$

4. Choosing a Utility Function

Find a parametric family of utility functions that possesses the relevant characteristics (such as risk aversion) previously specified for the decision maker.

Using the quantitative assessments, try to find a specific member of that family that is appropriate for the decision maker.

An example of monotonically increasing, decreasingly risk averse utility function:

$$u(x) = h + k(-e^{-ax} - be^{-cx}), \quad a, b, c, \text{and } k \geq 0$$

5. Checking for Consistency

Goals:

- to uncover discrepancies in the utility function
- to identify any intransitivities in the preferences
- to ensure that the risk preferences have been correctly modeled
- to ensure that the true preferences have been correctly encoded in the utility function

Expected Utility Calculation

Expected utility of a random outcome $\tilde{x}$:

$$E[u(\tilde{x})] = \int_{-\infty}^{\infty} u(x)f(x)dx$$

Determine the sensitivity of the decision to the parameters of the utility function.

Summary

1. Decision Making under Uncertainty
   - Alternate Approaches to Risky Choice Problem
   - Motivation for Using Utility Theory
2. Unidimensional Utility Theory
   - Fundamentals of Utility Theory
   - Qualitative Characteristics of Utility
3. Attitudes to Risk
   - Risk Averse and Risk Prone
   - Restricting the form of Utility Functions
   - Measuring Risk Aversion
4. Procedures for Assessing Utility Functions

Keeney, R. L. and H. Raiffa (1993). *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. Cambridge, UK, Cambridge University Press. Chapter 4.