Module 05
Probability Theory – An Overview

Dr. Jitesh H. Panchal

Decision Making in Engineering Design

School of Mechanical Engineering
Purdue University, West Lafayette, IN

Module Overview

1. Classical (Frequentist) Theory of Probability
2. Subjective Probability
   - Coherence and the Dutch Book
   - Measurement: Assessing Subjective Probabilities
3. Probability Calculus – A Quick Recap
   - Definitions
   - Bayes' Rule
   - Probability Distributions

Classical (Frequentist) Theory of Probability

Relative Frequency Theory of Probability

- "Probability" = Relative frequency of a property in a population.
- Probability is an empirical, measurable property of the actual physical world.
- 'Probability calculus' as an axiomatic mathematical tool for dealing with reality – just as arithmetic and geometry are.
- Individual probability statements attribute an empirical property to an empirical population.

Weatherford, R., 1982, *Philosophical Foundations of Probability Theory*, Routledge & Kegan Paul, London.

Relative Frequency Theory of Probability

Typically used in problems involving game of chance:

1. Tossing a coin
2. Rolling a die
3. Drawing cards from a deck
4. Picking the winning number in a lottery

Relative Frequency Theory of Probability

The problem of assigning probabilities to events is equivalent to counting how many outcomes are favorable to the occurrence of the event as well as how many are in the sample space, and then finding the ratio.

Let the sample space contain N distinct outcomes, $S = \{e_1, e_2, \ldots, e_N\}$
If the events are equally likely, then $p_1 = p_2 = \cdots = p_N$. Therefore, $p_i = P(\{e_i\}) = \frac{1}{N}$. It follows that

$$P(A) = \frac{n(A)}{N}$$

where $n(A)$ is the number of outcomes in A.

Subjective Probability

Probability: A Subjective Interpretation

In many cases, uncertainty is “in the mind”. Examples:

1. You have flipped a coin that has landed on the floor. Neither you nor anyone else has seen it. What is the probability that it is heads?
2. What is the probability that Oregon beat Stanford in their 1970 football game?
3. What is the probability that the coin that was flipped at the beginning of that game came up heads?
4. What is the probability that Millard Fillmore was president in 1850?

Subjective interpretation

Probability represents an individual's *degree of belief* that a particular outcome will occur.

The Subjectivist Theory of Probability (de Finetti and Savage)

- Probability is the degree of belief of a given person in a given proposition.
- Probabilities are best estimated by examining behavior, especially betting behavior.
- There are no objective probabilities.
- An event has no unique probability. Each individual is logically free to set his/her own values of probability.
- The probability beliefs of a rational individual must be consistent and governed by the calculus of probability.

All probability phenomena are personal, subjectivist, degrees of beliefs or expectation.

Weatherford, R., 1982, *Philosophical Foundations of Probability Theory*, Routledge & Kegan Paul, London.

The Subjectivist Theory of Probability

Coherence and Measurement

1. **Coherence**: Not just any degree of belief is admissible. Each person's set of degrees of belief in various propositions must agree with or conform to each other in a certain way.
2. **Measurement**: 'Degree of belief' is only meaningful when one has specified how it is to be measured or obtained.

Weatherford, R., 1982, *Philosophical Foundations of Probability Theory*, Routledge & Kegan Paul, London.

Coherence

Subjective probabilities must obey the same postulates and laws as so-called objective probabilities.

- If an individual's subjectively assessed probabilities obey the probability postulates, the person is said to be *coherent*.
- If a person is not coherent, then it is possible to set up a *Dutch book* against him or her.

Dutch book: series of bets that guarantees your opponent will lose and you will win.

Dutch Book: Example

Consider a person whose subjective probabilities are:

- Probability of 0.4 that Lakers will win
- Probability of 0.5 that Celtics will win

The person should be willing to agree to the following bets:

Bet 1 He wins $40 if the Lakers lose.
You win $60 if the Lakers win.

Bet 2 You win $50 if the Celtics win.
He wins $50 if the Celtics lose.

Note: His expected payoffs are the same ($=0$) in both bets.

The Dutch Book Theorem

The Dutch Book Theorem

A decision maker's degree of belief satisfy the probability axioms if and only if no Dutch Book can be made against her.

Measurement: Assessing Subjective Probabilities

The measurement process is based on the insight that the degree to which a decision maker believes something is closely linked to his/her behavior.

Three basic methods of measurement:

1. Asking the decision maker: “What is your belief regarding the probability that event such and such will occur?”
2. Ask about the bets that the decision maker is willing to place.
3. Using thought experiments with a reference lottery.

Assessing Subjective Probabilities – Behavioral Approach

Choose 1 or 2

1. If the coin lands heads up you win a sports car; otherwise you win nothing.
2. If the coin *does not* land heads up you win a sports car; otherwise you win nothing.

Peterson, M. (2009). *An Introduction to Decision Theory*, Cambridge University Press, Cambridge, UK.

Assessing Subjective Probabilities – Second Approach

Basic idea: To find a specific amount to win or lose such that the decision maker is indifferent about which side of the bet to take (i.e, the expected value is the same for both sides of the bet).

Example

Bet 1: Win \$X if the Lakers win; Lose \$Y if the Lakers lose.
Bet 2: Lose \$X if the Lakers win; Win \$Y if the Lakers lose.

If the decision maker is indifferent between Bets 1 and 2, then their expected values must be equal:

$$X[P_w] - Y[1 - P_w] = -X[P_w] + Y[1 - P_w]$$

where P_w is the probability that Lakers win. Therefore, the estimated subjective probability is:

$$P_w = \frac{Y}{X + Y}$$

Assessing Subjective Probabilities – Third Approach

Thought experiment strategy.

Compare the two lotteries

1. Win Prize A if Lakers win. Win prize B if the Lakers lose.

2. *Reference lottery:* Win Prize A with known probability p. Win Prize B with the probability $(1 - p)$

Adjust p in the reference lottery until the decision maker is indifferent between the lotteries.

The subjective probability that the Lakers win must be the p that makers the decision maker indifferent.

Assessing Subjective Probabilities through Decomposition

P(Stock Price Up) = P(Stock Price Up | Market Up) P(Market Up)
+ P(Stock Price Up | Market Not Up) P(Market Not Up)

$$P(A) = P(A|L,N)P(L,N) + P(A|L',N)P(L',N)$$
$$+P(A|L,N')P(L,N') + P(A|L',N')P(L',N')$$

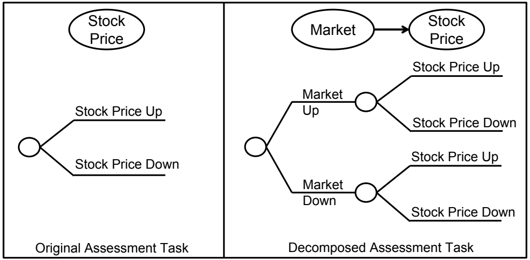


Figure: 8.14 on page 286 (Clemen, 1996)

Virtues of Subjectivist Theories of Probability

1. It accommodates talk of the probability of individual events.
2. It expands the field of possible applications of Bayes' theorem.
3. It has made contributions to the growth of decision theory and psychological knowledge.
4. It shows clearly why we must act on the probability calculus to be rational.

Probability Calculus – A Quick Recap

Probability Calculus

The probability calculus is an axiomatised theory — All true (mathematical) statements about probability can be derived from a small set of basic principles, or axioms.

Kolmogorov's Axioms, 1993

1. Every probability is a real number between 0 and 1.
2. The probability of the entire sample space is 1.
3. It two events are mutually exclusive, then the probability that one of them will occur equals the probability of the first plus the probability of the second.

Sample Space

Definition (Sample Space)

The set of all possible outcomes of an experiment is called the **sample space**, denoted by S.

Example: Tossing two coins: $S = \{\text{HH, HT, TH, TT}\}$

Definitions

Definition (Event)

An event is a subset of the sample space S. If A is an event, then A has occurred if it contains the outcome that occurred.

Example: The subset $A = \{\text{HH, HT, TH}\}$ contains the outcomes that correspond to the event of obtaining at least one head. If one of the outcomes in A occurs, then we say that the event A has occurred.

If A and B are events, so are A-or-B, A-and-B, not-A, and not-B.

Definitions and Basic Rules

Definition (Mutually Exclusive Events)

Events $A_1, A_2, \ldots$ are said to be **mutually exclusively** if they are pairwise mutually exclusive. That is, if $A_i \cap A_j = \varnothing$ whenever $i \neq j$.

Definition of Probability

Definition (Probability)

For a given experiment, S denotes the sample space and $A, A_1, A_2, \ldots$ represent possible events. A set function that associates a real value $P(A)$ with each event A is called a **probability set function**, and $P(A)$ is called the probability of A, if the following properties are satisfied:

1. $0 \leq P(A)$ for every A
2. $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$
3. $P(S) = 1$

if $A_1, A_2, \ldots$ are pairwise mutually exclusive events.

Consequences of the Definition of Probability

1. The null event has a probability of zero, i.e.,

$$P(\varnothing) = 0$$

2. For mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$

Complement and Union

Theorem

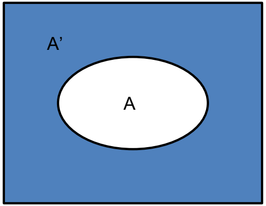
If A is an event and A' is its complement, then

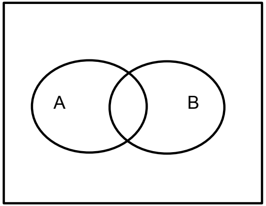
$$P(A) = 1 - P(A')$$

Theorem

For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$





Boole's and Bonferroni's Inequalities

Theorem (Boole's Inequality)

If $A_1, A_2, \ldots$ is a sequence of events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

Theorem (Bonferroni's Inequality)

If $A_1, A_2, \ldots$ is a sequence of events, then

$$P\left(\bigcap_{i=1}^{k} A_i\right) \geq 1 - \sum_{i=1}^{\infty} P(A_i')$$

Conditional Probability

Definition (Conditional Probability)

The conditional probability of an event A, given the event B, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

if $P(B) \neq 0$.

	B	B'	Totals
A	15	5	20
A'	45	35	80
Totals	60	40	100

$$P(A) = \frac{20}{100} = 0.20$$

$$P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{15}{60} = 0.25$$

Conditional Probability (contd.)

Theorem (Multiplication Theorem)

For any events A and B,

$$P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$$

Definition (Marginal Probabilities)

The probabilities, $P(A)$, $P(A')$, $P(B)$, $P(B')$ which only deals with one of the events is referred to as the marginal marginal probabilities (because they appear in the margins of the table).

Independence

Definition (Independent Events)

The events A and B are called independent events if

$$P(A \cap B) = P(A)P(B)$$

Otherwise, A and B are called dependent events.

Theorem

Events A and B are independent if and only if either of the following holds:

$$P(A|B) = P(A) \qquad P(B|A) = P(B)$$

Total Probability of an Event

If $B_1, B_2, \ldots, B_k$ are mutually exclusive and exhaustive (i.e., $B_1 \cup B_2 \cup \cdots \cup B_k = S$), then

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_k)$$

Theorem (Law of Total Probability)

If $B_1, B_2, \ldots, B_k$ are mutually exclusive and exhaustive, then for any event A,

$$P(A) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$$

Special case:

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B') \\ &= P(B)P(A|B) + P(B)P(A|B') \end{aligned}$$

Bayes' Theorem

Theorem

If $B_1, B_2, \ldots, B_k$ are mutually exclusive and exhaustive, then for each $j = 1, 2, \ldots, k$,

$$P(B_j|A) = \frac{P(B_j)P(A|B_j)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$$

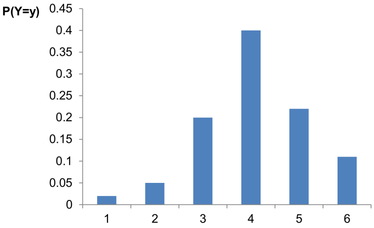
Special case:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B')P(B')}$$

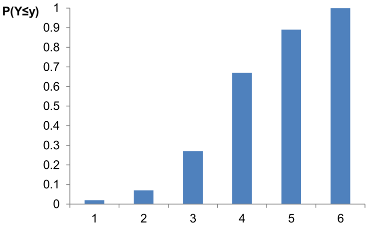
Discrete Probability Distributions

The set of probabilities associated with all possible outcomes of an uncertain quantity is called its *probability distribution*.
For discrete variables:

1. *Probability mass function* lists the probabilities for each possible discrete outcome.
2. *Cumulative distribution* gives the probability that an uncertain quantity is less than or equal to a specific value: $P(X \leq x)$



Probability Mass Function



Cumulative Distribution Function

Expected Value and Standard Deviation

Definition (Expected Value)

A discrete uncertain quantity's expected value is the probability-weighted average of its possible values.

$$E(X) = \sum_{i=1}^{n} x_i P(X = x_i)$$

Definition (Variance)

$$\begin{aligned} Var(X) = \sigma_X^2 &= \sum_{i=1}^{n} [x_i - E(X)]^2 P(X = x_i) \\ &= E[X - E(X)]^2 \end{aligned}$$

Continuous Variables

1. The probability of a particular value occurring is zero (Why?).

$$P(X = x) = 0$$

2. Instead, consider interval probabilities of the form:

$$P(a \leq X \leq b)$$

Probability Density Function (PDF): $f(x)$
Cumulative Distribution Function (CDF): $F(x) = P(X \leq x)$

$$P(a \leq X \leq b) = F(b) - F(a)$$

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

$$f(x) = \frac{d}{dx} F(x)$$

Continuous Probability Distributions – Example

Example: Consider a Cumulative Distribution Function (CDF):

$$F(x) = P[X \leq x] = cx$$

$$P[0 \leq X \leq 5] = 1$$

Expected Value and Variance

$$E(X) = \int_{x^-}^{x^+} x f(x) dx$$

$$\text{Var}(X) = \sigma_X^2 = \int_{x^-}^{x^+} [x - E(X)]^2 f(x) dx$$

where x^- and x^+ represent the lower and upper bounds for the uncertain quantity X.

Summary

1. Classical (Frequentist) Theory of Probability
2. Subjective Probability
   - Coherence and the Dutch Book
   - Measurement: Assessing Subjective Probabilities
3. Probability Calculus – A Quick Recap
   - Definitions
   - Bayes' Rule
   - Probability Distributions

References

1. Bain, L. J., Engelhardt, M. (1992). *Introduction to Probability and Mathematical Statistics*. Belmont, CA, Brooks/Cole, Cengage Learning.
2. Clemen, R. T. (1996). *Making Hard Decisions: An Introduction to Decision Analysis*. Belmont, CA, Wadsworth Publishing Company. Chapters 7, 8.
3. Howard, R.A. and Abbas, A. E. (2016). *Foundations of Decision Analysis*. Pearson Education. Chapter 6.
4. Peterson, M. (2009). *An Introduction to Decision Theory*, Cambridge University Press, Cambridge, UK. Chapters 6, 7.
5. Weatherford, R. (1982). *Philosophical Foundations of Probability Theory*, Routledge & Kegan Paul, London. ISBN: 0-7100-9002-1.