Bounds for the Condition Numbers of Spatially-variant Convolution Matrices in Image Restoration Problems

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Abstract: This paper presents theoretical results on the condition numbers of spatially variant convolution matrices. We show that they are bounded by the condition numbers of its circulant components.

1. Introduction

Image restoration is an inverse problem where the goal is to recover a sharp image from a blurry and noisy observation. Using the classical shift-invariant imaging system model [1], the input-output relationship is given by

$$\mathbf{g} = \mathbf{Hf} + \eta,$$

where $\mathbf{f}$ is a vector denoting the unknown (potentially sharp) image, $\mathbf{g}$ is the observed blurry and noisy image, η is the noise vector, and $\mathbf{H}$ is a matrix that models the blur (convolution matrix).

If the blur is spatially invariant, meaning that all pixels of the image $\mathbf{f}$ are identically blurred, then the matrix $\mathbf{H}$ has a block-circulant-with-circulant-block (BCCB) structure [2]. In this case, $\mathbf{H}$ can be diagonalized using Fourier transforms as $\mathbf{H} = \mathbf{F}^H \Lambda \mathbf{F}$, where $\mathbf{F}$ is the discrete Fourier transform (DFT) matrix, Λ is the (diagonal) eigenvalue matrix, and $(\cdot)^H$ is the conjugate transpose of the argument. However, if the blur is *spatially variant*, $\mathbf{H}$ is not diagonalizable using DFT matrices, thus making the spectral analysis of $\mathbf{H}$ difficult.

In the literature, studies on spatially variant convolution matrices are limited. Most of the work is on developing a restoration algorithm instead of analyzing the properties of such matrices. Nagy and O'Leary [5] proposed a singular value decomposition (SVD) method to approximate spatially variant convolution matrices so that the restoration problem can be solved using conjugate gradient. Later, Kamm and Nagy [6] extended the idea to separable spatially variant convolution matrices. However, analysis on the spatially variant convolution matrix was not pursued.

The goal of this paper is to study the eigenvalues of $\mathbf{H}^H\mathbf{H}$, which play a vital role in solving the normal equation for the least-squares minimization often used in image restoration:

$$\underset{\mathbf{f}}{\text{minimize}} \ \ \|\mathbf{Hf} - \mathbf{g}\|^2 + \alpha\|\mathbf{Df}\|^2, \tag{1}$$

where α is a regularization parameter and $\mathbf{D}$ is a linear transformation applied to $\mathbf{f}$. We estimate the upper and lower bounds on the largest and smallest eigenvalues of $\mathbf{H}^H\mathbf{H}$, and hence the condition number of $\mathbf{H}^H\mathbf{H}$. We are particularly interested in spatially-variant convolution matrices arising from spherical aberration and defocus with different object depths. As each pixel is approximately blurred by a Gaussian point spread function (PSF), the eigenvalues of the PSFs are *nonnegative*. Fig. 1 shows the simulation involving a spherical aberration and its restoration result.

2. Upper and Lower Bounds

For simplicity, the derivations presented in this paper are based on one-dimensional signals, but the results can be extended to the two-dimensional case. All matrices are assumed to be of size $n \times n$ and have real entries. Since real matrices can have complex eigenvalues, by the smallest (or largest) eigenvalue we mean the eigenvalue with the smallest (or largest) complex modulus. The smallest and largest eigenvalues of a matrix $\mathbf{H}$ are denoted as $\lambda_{\min}(\mathbf{H})$ and $\lambda_{\max}(\mathbf{H})$, respectively. Also, $|\Lambda|$ denotes the element-wise complex modulus of Λ, and Λ^* denotes the element-wise complex conjugate of Λ.

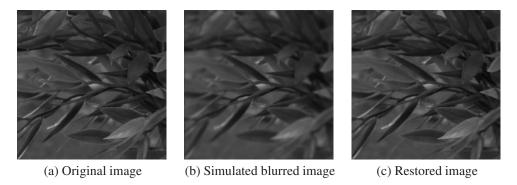(a) Original image   (b) Simulated blurred image   (c) Restored image

Fig. 1. An illustration of spherical aberration and restoration. The spatially variant convolution matrix used in (b) is generated using [3]. The restoration is performed using a modification of the least-squares total variation minimization [4].

Definition 1. *[7] A matrix* $\mathbf{H}$ *is* circulant *if each row is a circular shift of its preceding row. If* $\mathbf{h}$ *is a column vector, we use* $\texttt{CircMtx}(\mathbf{h},k)$ *to denote the circulant matrix generated by* $\mathbf{h}$, *with* $\mathbf{h}$ *being put in the k-th column.*

Our approach to analyzing a spatially-variant convolution matrix $\mathbf{H}$ is to consider all circulant matrices generated by the columns of $\mathbf{H}$. Thus, we define the *circulant components* of $\mathbf{H}$ as follows.

Definition 2. *Partitioning* $\mathbf{H}$ *into n column vectors as* $\mathbf{H} = (\mathbf{h}_1|\mathbf{h}_2|\cdots|\mathbf{h}_n)$, *where* $\mathbf{h}_k$ *denotes the k-th column of* $\mathbf{H}$, *we define the k-th* circulant component *of* $\mathbf{H}$ *as* $\texttt{CircMtx}(\mathbf{h}_k,k)$, *denoted by* $\mathbf{H}_k$.

The following main result states that the smallest eigenvalue of $\mathbf{H}^H\mathbf{H}$ is lower-bounded by the smallest eigenvalue among all of the circulant components of $\mathbf{H}$.

Theorem 1. *Let* $\mathbf{H}$ *be a spatially-variant convolution matrix, and let* $\mathbf{H}_1$, $\mathbf{H}_2,\ldots,\mathbf{H}_n$ *be the circulant components of* $\mathbf{H}$. *If all eigenvalues of* $\mathbf{H}_k$ *are nonnegative, the smallest eigenvalue of* $\mathbf{H}^H\mathbf{H}$ *is bounded from below by*

$$|\lambda_{\min}(\mathbf{H}^H\mathbf{H})| \geq \min_k \left\{ |\lambda_{\min}(\mathbf{H}_k)|^2 \right\},$$

where $|\lambda_{\min}(\mathbf{H}_k)|$ *is the smallest eigenvalue of* $\mathbf{H}_k$.

Proof. Let $\mathbf{E}_k = \text{diag}\{0, \ldots, 1, \ldots, 0\}$ be a diagonal matrix with the (k,k)-th entry being 1. Using $\mathbf{E}_k$, we can express $\mathbf{H}$ as a sum of its circulant components as $\mathbf{H} = \mathbf{H}_1\mathbf{E}_1 + \ldots + \mathbf{H}_n\mathbf{E}_n$. Taking the conjugate transpose and multiplying with $\mathbf{H}$ yields

$$\mathbf{H}^H\mathbf{H} = \sum_{i,j} \mathbf{E}_i^H \mathbf{H}_i^H \mathbf{H}_j \mathbf{E}_j. \tag{2}$$

Let $\mathbf{u}$ be the eigenvector associated with $\lambda_{\min}(\mathbf{H}^H\mathbf{H})$. By multiplying $\mathbf{u}^H$ and $\mathbf{u}$ on both sides of Eqn. (2), the (i,j)-th term in the sum is

$$\mathbf{u}^H (\mathbf{E}_i^H \mathbf{H}_i^H \mathbf{H}_j \mathbf{E}_j)\mathbf{u} = \mathbf{u}^H \mathbf{E}_i^H \mathbf{F}^H \Lambda_i^* \mathbf{F}\mathbf{F}^H \Lambda_j \mathbf{F}\mathbf{E}_j\mathbf{u}$$
$$\geq \mathbf{u}^H \mathbf{E}_i^H \mathbf{F}^H \left(|\lambda_{\min}(\Lambda_i^*)||\lambda_{\min}(\Lambda_j)|\mathbf{I}\right)\mathbf{F}\mathbf{E}_j\mathbf{u} = \mathbf{u}^H \mathbf{E}_i^H \mathbf{F}^H \left(|\lambda_{\min}(\mathbf{H}_i)||\lambda_{\min}(\mathbf{H}_j)|\mathbf{I}\right)\mathbf{F}\mathbf{E}_j\mathbf{u}$$
$$= |\lambda_{\min}(\mathbf{H}_i)||\lambda_{\min}(\mathbf{H}_j)|\mathbf{u}^H \mathbf{E}_i^H \mathbf{F}^H \mathbf{F}\mathbf{E}_j\mathbf{u} = \begin{cases} |\lambda_{\min}(\mathbf{H}_i)|^2|u_i|^2, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

because $\mathbf{F}^H\mathbf{F} = \mathbf{F}\mathbf{F}^H = \mathbf{I}$, and $\mathbf{E}_i^H\mathbf{E}_j = \mathbf{0}$ if $i \neq j$. Here, u_i is the i-th element of $\mathbf{u}$. Note that the first inequality holds because the eigenvalues of a Gaussian point spread function are real and nonnegative.

Therefore, the smallest eigenvalue of $\mathbf{H}^H\mathbf{H}$ is

$$|\lambda_{\min}(\mathbf{H}^H\mathbf{H})| \geq \sum_{i=1}^{n} |\lambda_{\min}(\mathbf{H}_i)|^2|u_i|^2 \geq \left(\min_i\{|\lambda_{\min}(\mathbf{H}_i)|^2\}\right)\sum_{i=1}^{n} |u_i|^2 = \min_i\{|\lambda_{\min}(\mathbf{H}_i)|^2\},$$

because the eigenvector $\mathbf{u}$ has unit norm so that $\sum_{i=1}^{n} |u_i|^2 = 1$. Thus, we have $|\lambda_{\min}(\mathbf{H}^H\mathbf{H})| \geq \min_i \left\{|\lambda_{\min}(\mathbf{H}_i)|^2\right\}$. $\square$

By flipping the inequality signs in the above proof, we have a similar result for the maximum eigenvalue of $\mathbf{H}^H\mathbf{H}$.

Theorem 2. *Let* $\mathbf{H}$ *be a spatially-variant convolution matrix, and let* $\mathbf{H}_1$, $\mathbf{H}_2, \ldots, \mathbf{H}_n$ *be the circulant components of* $\mathbf{H}$*. If all eigenvalues of* $\mathbf{H}_k$ *are nonnegative, the largest eigenvalue of* $\mathbf{H}^H\mathbf{H}$ *is upper-bounded by*

$$|\lambda_{\max}(\mathbf{H}^H\mathbf{H})| \leq \max_k \left\{ |\lambda_{\max}(\mathbf{H}_k)|^2 \right\},$$

where $|\lambda_{\max}(\mathbf{H}_k)|$ *is the largest eigenvalue of* $\mathbf{H}_k$*.*

Using Theorems 1 and 2, we derive a corollary for the condition number of $\mathbf{H}^H\mathbf{H}$.

Corollary 1. *Suppose* $\mathbf{H}$ *is a spatially-variant matrix, and let* $\mathbf{H}_1$, $\ldots$, $\mathbf{H}_n$ *be the circulant components of* $\mathbf{H}$*. The condition number of* $\mathbf{H}^H\mathbf{H}$ *is bounded from above by*

$$\mathrm{cond}\left(\mathbf{H}^H\mathbf{H}\right) \leq \frac{\max_k \left\{ |\lambda_{\max}(\mathbf{H}_k)|^2 \right\}}{\min_k \left\{ |\lambda_{\min}(\mathbf{H}_k)|^2 \right\}}, \tag{3}$$

where $\lambda_{\min}(\mathbf{H}_k)$ *and* $\lambda_{\max}(\mathbf{H}_k)$ *are the minimum and maximum eigenvalues of* $\mathbf{H}_k$*, respectively.*

Corollary 1 implies that a spatially-variant blur (e.g., spherical aberration) can be interpreted as a set of spatially-invariant blurs. The condition number of $\mathbf{H}^H\mathbf{H}$ is never larger than the upper bound given in Eqn. (3). Moreover, in the spherical aberration case where the blur consists of a collection of Gaussian PSFs, the bound in Eqn. (3) can be computed using the PSFs with the largest and smallest variance. The following corollary is useful in analyzing the solution to the regularized least-squares problem expressed in Eqn. (1).

Corollary 2. *The smallest eigenvalue of* $\mathbf{H}^H\mathbf{H} + \alpha\mathbf{D}^H\mathbf{D}$ *is bounded by*

$$\left|\lambda_{\min}\left(\mathbf{H}^H\mathbf{H} + \alpha\mathbf{D}^H\mathbf{D}\right)\right| \geq \min_k \left\{ \min_j \left\{ |\lambda_j^{\mathbf{H}_k}|^2 + \alpha|\lambda_j^{\mathbf{D}}|^2 \right\} \right\},$$

where $\lambda_j^{\mathbf{H}_k}$ *is the j-th eigenvalue of* $\mathbf{H}_K$ *and* $\lambda_j^{\mathbf{D}}$ *is the j-th eigenvalue of* $\mathbf{D}$*.*

Proof. Let $\mathbf{u}$ be the eigenvector associated with the minimum eigenvalue of $\mathbf{H}^H\mathbf{H} + \alpha\mathbf{D}^H\mathbf{D}$. It can be shown that $\left|\lambda_{\min}\left(\mathbf{H}^H\mathbf{H} + \alpha\mathbf{D}^H\mathbf{D}\right)\right| = \left|\mathbf{u}^H\left(\sum_{i,j=1}^n \mathbf{E}_i^H\mathbf{H}_i^H\mathbf{H}_j\mathbf{E}_j + \alpha\mathbf{D}^H\mathbf{D}\right)\mathbf{u}\right| \geq \sum_{k=1}^n \lambda_{\min}\left(|\Lambda^{\mathbf{H}_k}|^2 + \alpha|\Lambda^{\mathbf{D}}|^2\right)\mathbf{u}^H\mathbf{E}_i^H\mathbf{F}^H\mathbf{F}\mathbf{E}_j\mathbf{u} \geq \min_k \left\{ \min_j \left\{ |\lambda_j^{\mathbf{H}_k}|^2 + \alpha|\lambda_j^{\mathbf{D}}|^2 \right\} \right\}.$ $\square$

3. Conclusion

Eigenvalues of spatially-variant convolution matrices are studied. If all their circulant components have nonnegative eigenvalues, the smallest eigenvalue is lower bounded by the minimum eigenvalue among all its circulant components. Consequently, bounds on condition numbers can be derived. We also derived the bounds on the eigenvalues of the normal equation matrix arises from least-squares image restoration formulation.

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