Normalization

Question: Will normalization of columns of $A$ affect the performance of OMP?

Let $A$ be a matrix. Let $\tilde{A}$ be the normalized matrix
\[ \tilde{A} = AW \]
so that $\|\tilde{a}_j\|_2 = 1$.

Theorem (3.1)
The OMP produces the same support $S^k$ when using either $A$ or $\tilde{A}$.

Proof: Consider the sweep:
\[ e(j) = \| r^{k-1} \|_2^2 - \left( \frac{a_j^T r^{k-1}}{\| a_j \|_2} \right)^2 \]
\[ = \| r^{k-1} \|_2^2 - (\tilde{a}_j^T r^{k-1})^2 \]
\[ = \| r^{k-1} \|_2^2 - \left( \frac{\tilde{a}_j^T r^{k-1}}{\| \tilde{a}_j \|_2} \right)^2 . \]

So using $a_j$ or $\tilde{a}_j$ will lead to the same $e(j)$, and hence
\[ j^* = \arg \min_j e(j) \]
is the same for both $a_j$ and $\tilde{a}_j$. 
For the LS step, since
\[ x_s = (A_s^T A_s)^{-1} A_s^T b, \]
So \[ r^k = b - A_s x_s \]
\[ = b - A_s (A_s^T A_s)^{-1} A_s^T b \]
\[ = (I - A_s (A_s^T A_s)^{-1} A_s^T ) b \]
\[ = (I - A_s W_s W_s^T (A_s^T A_s)^{-1} W_s^T W_s A_s^T ) b \]
\[ = (I - A_s W_s (W_s A_s A_s^T A_s W_s)^{-1} W_s A_s^T ) b \]
\[ = (I - \tilde{A}_s (\tilde{A}_s^T \tilde{A}_s)^{-1} \tilde{A}_s^T ) b = \tilde{r}^k \]

Therefore, the residue is unchanged.
Convex Relaxation Algorithm

Idea: Convert \((P_0)\) into a sequence of easy problems.

Focuss Algorithm

Current approximate solution \(X_{k-1}\).

let \(X_{k-1} = \text{diag}(1, |x_{k-1}|^p)\).

Then, consider

\[
\| X_{k-1}^{-1} \times \|^2_2
\]

\[
\approx \| (\frac{1}{|x_i|^2} \cdot \cdots \cdot \frac{1}{|x_m|^2})(x_i) \|^2_2
\]

\[
= \| x \|_2^{2p-2}
\]

if we choose \(p = 1\), then \(\frac{1}{p} = \frac{1}{2}\).

(0 < p ≤ 1)

So, if \(\frac{1}{p} = 1 - \frac{p}{2}\), then \(\| X_{k-1}^{-1} \times \|^2_2 \approx \| x \|^p_2\).

Focuss Algorithm solves

\[
(M_k) \quad \min_x \| X_{k-1}^{-1} x \|^2_2 \quad \text{subject to } Ax = b.
\]

Solution of \((M_k)\):

\[
\dot{L}(x) = \| X_{k-1}^{-1} x \|^2_2 + x^T(b - Ax)
\]

\[
\frac{3}{3x}L = 0 \implies x_k = \frac{1}{2} X_{k-1}^{-1} A^T \lambda
\]

\[
\frac{3}{3\lambda}L = 0 \implies \lambda = 2(A X_{k-1}^{-1} A^T)^{-1} b
\]

So \(x_k = X_{k-1}^{-1} A^T (A X_{k-1}^{-1} A^T)^+ b\).
Two remarks for Focuss:

1. It guarantees convergence to a fixed point, not the global minimum. Instead, it may get stuck on a steady-state solution.

2. We need to initialize focuss at a non-zero entry. Otherwise it will just stay at zero.

Basis Pursuit

\[
(P_1) \quad \min_x \|x\|_1, \\
\text{subject to } Ax = b.
\]

Possible solutions:
- linear programming
- interior point method
- CVX
- L1 - LS
- SparCo
- L1 - magic
- SPAMS
- etc.
Performance Guarantee of OMP

Goal: To analyze the performance of OMP, and determine conditions under which OMP will return the sparsest solution.

Setting: Assume that the system $A x = b$ has a sparse solution $x$ with $k_0$ non-zeros, i.e., $\|x\|_0 = k_0$.

Assume that $k_0 < \text{spark}(A)/2$.

Consider a two-ortho system:

$$A = \begin{bmatrix} \Phi & \Phi \end{bmatrix},$$

so that $b$ is created by the first $k_p$ columns of $\Phi$, and the first $k_\Phi$ columns of $\Phi$, such that $k_p + k_\Phi = k_0$:

$$b = \sum_{i=1}^{k_p} x_i \psi_i + \sum_{i=1}^{k_\Phi} x_i \phi_i$$

Let $S_p$ and $S_\Phi$ be the sets of support indices.