**Mutual Coherence (General Matrix)**

The mutual coherence of a matrix $A$ is

$$\mu(A) = \max_{1 \leq i, j \leq m} \frac{|a_i^T a_j|}{\|a_i\|_2 \|a_j\|_2}.$$ 

**Lemma 2.1**

For any matrix $A$,

$$\text{spark}(A) \geq 1 + \frac{1}{\mu(A)}.$$

**Proof** WLOG Assume columns of $A$ are normalized so that $\|a_i\|_2 = 1$.

Let $G = A^T A$.

It holds that

1. $G_{kk} = 1$
2. $|G_{kj}| = |a_i^T a_j| \leq \mu(A)$.

Choose a $p \times p$ leading minor from $G$. Call this submatrix $H$. 
Gershgorin says if $H$ is diagonally dominant, then $H$ is non-singular. This will then imply that the $p$ columns are linearly independent.

Therefore to have $H$ being strictly diagonally dominant, we require that

$$|H_{ii}| > \sum_{j \neq i} |H_{ij}|$$

$$\implies 1 > \sum_{j \neq i} |H_{ij}| \geq (p-1)\mu(A).$$

So, if we have $1 > (p-1)\mu(A)$,

ie. $p < 1 + \frac{1}{\mu(A)}$.

then $H$ is non-singular.

However, as soon as we reach

$$p = 1 + \frac{1}{\mu(A)},$$

then we will start to get dependent columns, possibly. So $1 + \frac{1}{\mu(A)}$ is the smallest number that can possibly cause dependence. So

$$\text{spark}(A) \geq 1 + \frac{1}{\mu(A)}.$$

Note: this is a lower bound. It just says $\text{spark}(A)$ cannot be smaller than $1 + \frac{1}{\mu(A)}$. 
Gershgorin Circle Theorem

For a square matrix $A \in \mathbb{R}^{n \times n}$, all eigenvalues $\lambda_1, \ldots, \lambda_n$ lie in the union of $n$ discs, each with center $a_{ii}$, radius $r_i = \sum_{j \neq i} |a_{ij}|$.

**Pf:** Let $x$ be an eigenvector of $A$, with $\lambda$ be the eigenvalue. So, $Ax = \lambda x$.

Let $i^* = \text{argmax}_i |x_i|$, so that $|x_{i^*}| \geq |x_i|$ for all $i$.

Consider the $j$th column of $A x$:

$$\sum_{j=1}^{n} a_{i^*j} x_j = \lambda x_{i^*}$$

$$\Rightarrow \sum_{j \neq i^*} a_{i^*j} x_j = x_{i^*}(\lambda - a_{i^*i^*})$$

$$\Rightarrow \lambda - a_{i^*i^*} = \frac{1}{x_{i^*}} \sum_{j \neq i^*} a_{i^*j} x_j$$

$$\Rightarrow |\lambda - a_{i^*i^*}| \leq \frac{\sum_{j \neq i^*} |a_{i^*j}| |x_j|}{|x_{i^*}|}$$

$$\leq \sum_{j \neq i^*} |a_{i^*j}|.$$

So $\lambda \in \text{disc}(a_{i^*i^*}, \sum_{j \neq i^*} |a_{i^*j}|)$

$$\subseteq \text{Union of all discs of } (a_{ii}, \sum_{j \neq i} |a_{ij}|)$$
Strictly diagonal dominant matrix

A matrix $A$ is strictly diagonal dominant if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \text{ for all } i.$$

Theorem

A strictly diagonal dominant matrix is non-singular.

Proof: If $A$ is singular, then $\exists$ eigenvalue $\lambda = 0$.

By Gershgorin,

$$|0 - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|$$

$$\Rightarrow |a_{ii}| \leq \sum_{j \neq i} |a_{ij}|.$$

Contradicts with the definition of

strictly diagonal dominant matrix.
Theorem 2.5 \( \text{Ax=b} \)

If a system has a solution \( x \) s.t.
\[ \|x\|_0 < \frac{1}{2} \left(1 + \frac{1}{\mu(A)}\right), \]
then \( x \) is necessarily the most sparse solution.

Proof. By theorem 2.4, \( x \) is the sparsest if
\[ \|x\|_0 < \text{spark}(A)/2 \]

So if \( \|x\|_0 < \frac{1}{2} \left(1 + \frac{1}{\mu(A)}\right) \)
then Lemma 1 implies that
\[ \|x\|_0 < \text{spark}(A)/2 \]
So \( x \) is the sparsest solution.