Uncertainty Principle for Sparse Problems

"Two-ortho" system:
\[ [\Phi, \Phi^T] x = b, \]
where \( \Phi \) and \( \Phi^T \) are orthonormal matrices.
Example, \( \Phi = I, \Phi^T = \text{Fourier} \).

Theorem (Textbook 2.1)

Let \( \alpha \) and \( \beta \) be such that
\[ b = \Phi \alpha, \quad b = \Phi^T \beta. \]
Then,
\[ \| \alpha \|_0 + \| \beta \|_0 \geq \frac{2}{\mu(A)}, \quad A = [\Phi, \Phi^T], \]
where \( \mu(A) \) def \( \max_{i,j} | \Phi_i^T \Phi_j | \).

Intuition:
\( \alpha \) and \( \beta \) are expansion coefficients of \( b \) with respect to \( \Phi \) and \( \Phi^T \). Theorem 2.1 says \( \alpha \) and \( \beta \) cannot be sparse simultaneously.
Property of $\mu(A)$ (the Mutual coherence)

For any two orthor system $A = [\Phi, \Phi]$, \[ \frac{1}{\sqrt{n}} \leq \mu(A) \leq 1 \]

Proof: " $\mu(A) \leq 1$ ".

Choose $\Phi = [\Phi_i, \Phi_j]$. Then \[ |\Psi_i^T \Phi_j|^2 \leq \|\Psi_i\|^2 \|\Phi_j\|^2 \] by Cauchy

\[ = 1 \quad \forall i, j . \]

So \[ \max_{i,j} |\Psi_i^T \Phi_j| \leq 1 . \]

" $\mu(A) \geq \frac{1}{\sqrt{n}}$ ".

Suppose that $\mu(A) < \frac{1}{\sqrt{n}}$. Then \[ \max_{i,j} |\Psi_i^T \Phi_j| < \frac{1}{\sqrt{n}} \quad \forall i, j \]

\[ \Rightarrow |\Psi_i^T \Phi_j| < \frac{1}{\sqrt{n}} \quad \forall i, j \]

Choose $\Phi_j = [0 \ldots 1 \ldots 0]_j^T$, i.e. $\Phi = I$.

Then \[ |\Psi_i^T \Phi_j| = |\Psi_{ij}| . \]
Since $|\psi_i^T \phi_j| < \frac{1}{\sqrt{n}} \quad \forall i, j,$

we have

$|\psi_{ij}| < \frac{1}{\sqrt{n}} \quad \forall i, j$

choose $\Phi = F$. Then

$\|\psi_i\|^2 = \sum_{j=1}^{n} \psi_{ij}^2 < \sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^2 = 1.$

Contradiction. So we must have $\mu(A) \geq \frac{1}{\sqrt{n}}$.

**Proof (of Theorem 2.1)**

WLOG assume $\|b\|_2 = 1$.

Then

$1 = b^T b$

$= \alpha^T \Phi \Phi \beta$

$= \sum_j \sum_i \alpha_i \beta_j \psi_i^T \phi_j$

$\leq \max_{i,j} |\psi_i^T \phi_j| \sum_i \sum_j |\alpha_i| |\beta_j|$

$= \mu(A) \|\alpha\|_1 \|\beta\|_1.$

So,

$\|\alpha\|_1 \|\beta\|_1 \geq \frac{1}{\mu(A)}.$
Can we find an upper bound on $\|x\|_1$?

Clearly, if for a given $b$, $\mathbf{A}$, the expansion coefficient is uniquely defined. However, suppose we know that $x$ contains $\eta$ non-zeros, then the unique $x$ is just one in the set

$$\Omega = \{ x \mid \|x\|_0 = \eta, \|x\|_2 = 1 \}.$$ 

Our goal is to find

$$\max_{x} \|x\|_1$$

s.t. $\|x\|_0 = \eta, \|x\|_2 = 1.$

(1)

If we can do that, then we will have

$$\square \geq \|x\|_1, \|x\|_1 \geq \frac{1}{\lambda(A)}$$

WLOG assume $x$ has non-zeros in the first $\eta$ entries. Then, the Lagrangian of (1) is

$$L(x, \lambda) = \|x\|_1 + \lambda (1 - \|x\|_2^2)$$

$$= \sum_{i=1}^{\eta} |x_i| + \lambda \left(1 - \sum_{i=1}^{\eta} |x_i|^2\right).$$
\[
\frac{2}{\delta |\alpha_i|^2} = 1 - 2\lambda |\alpha_i| = 0
\]

\[
\implies |\alpha_i| = \frac{1}{2\lambda}
\]

So,

\[
\sum_{i=1}^{q} |\alpha_i|^2 = 1 \implies \sum_{i=1}^{q} \left( \frac{1}{4\lambda^2} \right) = 1
\]

\[
\implies \frac{q}{4\lambda^2} = 1
\]

\[
\implies \lambda = \frac{\sqrt{q}}{2}
\]

Therefore,

\[
|\alpha_i| = \frac{1}{\sqrt{q}}, \quad \text{and so}
\]

\[
\|\alpha\|_1 = \sum_{i=1}^{q} \frac{1}{\sqrt{q}} = \sqrt{q} = \sqrt{\|\alpha\|_0}
\]

This gives us

\[
\sqrt{\|\alpha\|_0 \|\beta\|_0} \geq \frac{1}{\mu(A)}
\]

Since \(\sqrt{ab} \leq \frac{a+b}{2}\) (geometric mean \(\leq\) arithmetic mean),

we have

\[
\|\alpha\|_0 + \|\beta\|_0 \geq \frac{2}{\mu(A)}.
\]
Alternative (shorter) proof:

Since $\Phi$ and $\Psi$ are orthonormal, we have

$$\| b \|_2 = \| \alpha \|_2 = \| \beta \|_2 .$$

Suppose that the support of $\alpha$ is $I$, support of $\beta$ is $J$.

We know that $b = \Phi^\dagger \alpha$

$$= \sum_{i \in I} \alpha_i \psi_i .$$

Therefore,

$$| \beta_j |^2 = | \langle b , \phi_j \rangle |^2$$

$$= \left| \langle \sum_{i \in I} \alpha_i \psi_i , \phi_j \rangle \right|^2$$

$$= \left| \sum_{i \in I} \alpha_i \psi_i^T \phi_j \right|^2$$

$$\leq \| \alpha \|_2^2 \left| \sum_{i \in I} (\psi_i^T \phi_j)^2 \right|$$

$$\leq \| b \|_2^2 \sum_{i \in I} (\max_{i,j} | \psi_i^T \phi_j |^2)$$

$$= \| b \|_2^2 | I | \mu(A)^2 .$$

So

$$\sum_{j \in J} | \beta_j |^2 = \| b \|_2^2 | I | | J | \mu(A)^2 .$$

$$\Rightarrow \frac{\| \beta \|_2^2}{\| b \|_2^2} = \sqrt{| I | | J | \mu(A)^2} \geq \frac{1}{\mu(A)} .$$